

Analytical expansion to second order in mass in the planetary Hamiltonian

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Abstract

In this document, I perform a Lie serie expansion to second order in mass in the planetary Hamiltonian in order to obtain an analytical expression of the term in front of $\exp(\iota \mathbf{k} \cdot \boldsymbol{\lambda})$ with $\mathbf{k} \in \mathbb{Z}^3$ and $\boldsymbol{\lambda} \in (\mathbb{R}/2\pi\mathbb{R})^3$, at some degree in eccentricity and inclination. I extend the python package *celerier* of J.B. Delisle with a function **PerHam3pla** outputting that term at any degree in eccentricity and inclination.

1 Expansion to first order in mass

1.1 Framework

I consider a planetary system with n planets of masses m_i ($1 \leq i \leq n$) orbiting a star of mass m_0 . In this work, i is an index, I is an inclination and $\iota = \sqrt{-1}$. Working in canonical heliocentric variables, the Hamiltonian of the system can be written (Laskar and Robutel, 1995; Couturier, 2022, Eq. (2.53))

$$\mathcal{H}(\tilde{\mathbf{r}}, \mathbf{r}) = \sum_{i=1}^n \left(\frac{\tilde{\mathbf{r}}_i^2}{2\beta_i} - \frac{\mathcal{G}m_0m_i}{r_i} \right) + \sum_{i=1}^n \sum_{j=i+1}^n \left(\frac{\tilde{\mathbf{r}}_i \cdot \tilde{\mathbf{r}}_j}{m_0} - \frac{\mathcal{G}m_im_j}{|\mathbf{r}_i - \mathbf{r}_j|} \right), \quad (1)$$

where $\tilde{\mathbf{r}}_j = \beta_j \mathbf{v}_j$, with $\mathbf{v}_j$ the barycentric speed of planet j and $\mathbf{r}_j$ its heliocentric position. The reduced mass is $\beta_j = m_0m_j/(m_0 + m_j)$. I work in Poincaré coordinates defined as

$$\left. \begin{aligned} \Lambda &= \beta \sqrt{\mu a} \\ D &= \Lambda (1 - \sqrt{1 - e^2}) \\ H &= \Lambda \sqrt{1 - e^2} (1 - \cos I) \end{aligned} \right| \begin{aligned} \lambda &= M + \varpi, \\ -\varpi &= -\omega - \Omega, \\ &= -\Omega, \end{aligned} \quad (2)$$

where $(a, e, I, M, \omega, \Omega)$ are the usual elliptic elements and $\mu_j = G(m_0 + m_j)$, with G the gravitational constant. Alternatively, I will use the complex Poincaré variables defined by the canonical transformation

$$(D, H; -\varpi, -\Omega) \mapsto (x = \sqrt{D} \exp(\iota\varpi), y = \sqrt{H} \exp(\iota\Omega); -\iota\bar{x}, -\iota\bar{y}), \quad (3)$$

as well as the non-canonical variables $X = x\sqrt{2/\Lambda} = e \exp(\iota\varpi) + \mathcal{O}(e^3)$ and $Y = y/\sqrt{2\Lambda} = I \exp(\iota\Omega)/2 + \mathcal{O}(I^3) + \mathcal{O}(Ie^2)$.

1.2 Expansion in Poincaré coordinates

The first term of Eq. (1) is the Keplerian part of the Hamiltonian and corresponds to star–planet interactions. It can be written as a function of $\mathbf{\Lambda}$ only, and reads

$$\mathcal{H}_K(\mathbf{\Lambda}) = - \sum_{j=1}^n \frac{\beta_j^3 \mu_j^2}{2\Lambda_j^2}. \quad (4)$$

The second term of Eq. (1) is the perturbative part of the Hamiltonian and corresponds to planet–planet interactions. It is of characteristic size ε with respect to the Keplerian part, where ε is defined as

$$\varepsilon = \frac{m_1 + m_2 + \dots + m_n}{m_0}. \quad (5)$$

To emphasize the small size of the perturbation, I will write it $\varepsilon \mathcal{H}_P$. Hereafter, a quantity of order r in mass will be written with a factor ε^r . The perturbative part can be written

$$\varepsilon \mathcal{H}_P = \sum_{i=1}^n \sum_{j=i+1}^n \varepsilon \mathcal{H}_{i,j}, \quad (6)$$

which emphasizes the fact that it only contains interactions between two planets. In particular, at order 1 in mass, combinations of three or more angles will not appear. The perturbative part due to pair (i, j) can be expanded in X_i , X_j , Y_i and Y_j as (Laskar and Robutel, 1995, Eq. (73))

$$\varepsilon \mathcal{H}_{i,j} = \sum_{\mathbf{k} \in \mathbb{Z}^2} \left(\sum_{\mathbf{q} \in \mathbb{N}^8} \Xi X_i^{q_1} X_j^{q_2} \bar{X}_i^{q_3} \bar{X}_j^{q_4} Y_i^{q_5} Y_j^{q_6} \bar{Y}_i^{q_7} \bar{Y}_j^{q_8} \right) e^{\iota(k_1 \lambda_i + k_2 \lambda_j)}, \quad (7)$$

where Ξ depends on Λ_i and Λ_j only. In order to prevent an infinite number of terms in front of a given $e^{\iota(k_1 \lambda_i + k_2 \lambda_j)}$, this expression is truncated at some degree r in eccentricity and inclination, and only terms such that $|q_1| + \dots + |q_8| \leq r$ are retained. After this truncation, there are finitely many terms in front of $e^{\iota(k_1 \lambda_i + k_2 \lambda_j)}$, but a lot for large r . However, the symmetries of the Hamiltonian reveal that most of them are zero.

1.3 Extended d'Alembert rule

The first symmetry is the invariance around the z -axis. Rotating around the z -axis, that is, redefining the x and y axes, is equivalent to translate the angles of the problem by some constant angle ϑ . Injecting the transformation $(\lambda, \varpi, \Omega) \rightarrow (\lambda + \vartheta, \varpi + \vartheta, \Omega + \vartheta)$ into Eq. (7), shows that the Hamiltonian is invariant by rotation around the z -axis if, and only if, the tuples $\mathbf{q} = (q_1, \dots, q_8) \in \mathbb{N}^8$ and $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$ verify the d'Alembert rule (Chenciner and Laskar, 1989, Sect. 6.1)

$$k_1 + k_2 + q_1 + q_2 - q_3 - q_4 + q_5 + q_6 - q_7 - q_8 = 0. \quad (8)$$

Clearly, the dynamics of the system should not be affected by a redefinition of the x -axis, and so $\Xi = 0$ whenever the d'Alembert rule is not verified.

The other symmetry to consider is the symmetry with respect to the plane (x, y) . Changing z into $-z$, the ascending node become the descending node and vice-versa, and Ω is transformed into $\Omega + \pi$. Because ω is the angle between the ascending node and the periapsis, it is also transformed into $\omega + \pi$. The mean anomaly M is unchanged as it is defined from the periapsis which is unaffected by that symmetry. The angles $\varpi = \omega + \Omega$ and $\lambda = M + \varpi$ are also unchanged. Injecting this transformation into Eq. (7) yields an extension to the d'Alembert rule (Chenciner and Laskar, 1989, Sect. 6.1)

$$q_5 + q_6 + q_7 + q_8 = 0 \pmod{2}. \quad (9)$$

Since the dynamics should also not be affected by a symmetry with respect to the plane (x, y) , $\Xi = 0$ whenever the extension to the d'Alembert rule is not verified. The d'Alembert rule and its extension, along with a truncation in eccentricity and inclination at degree r , ensure that there are only a modest number of terms in front of $e^{\iota(k_1 \lambda_i + k_2 \lambda_j)}$ at a small degree r .

However, Eq. (7) overall still contains infinitely many terms. For example, each $e^{\iota(p \lambda_i - (p+1) \lambda_j)}$ contains terms of degree 1 for $p \in \mathbb{Z}$. In practice, only the terms $e^{\iota(k_1 \lambda_i + k_2 \lambda_j)}$ where $k_1 \lambda_i + k_2 \lambda_j$ is a slow angle are retained, the others being discarded (averaged upon). Even when fast angles are retained, a finite number of terms is assured using the fact that Ξ goes to 0 when (k_1, k_2) goes to infinity (see Table 1).

1.4 Expression as a function of the Laplace coefficients

The coefficients $\Xi(\Lambda_i, \Lambda_j)$ appearing in Eq. (7) can be written as a function of the Laplace coefficients $b_{s/2}^{(k)}(\alpha_{ij})$ where s and k are integers (k being odd), $\alpha_{ij} = a_i/a_j$ and¹ $b_{s/2}^{(k)}(\alpha)$ is defined as

$$b_{s/2}^{(k)}(\alpha) = \frac{2}{\pi} \int_0^\pi (1 - 2\alpha \cos \theta + \alpha^2)^{-s/2} \cos k\theta \, d\theta. \quad (10)$$

Using the recurrence relation (Brouwer and Clemence, 1961, p. 501)

$$b_{s/2}^{(k)}(\alpha) = \frac{2k-2}{2k-s} (\alpha + \alpha^{-1}) b_{s/2}^{(k-1)}(\alpha) - \frac{2k+s-4}{2k-s} b_{s/2}^{(k-2)}(\alpha), \quad (11)$$

and the fact that $b_{s/2}^{(-k)}(\alpha) = b_{s/2}^{(k)}(\alpha)$, it is easy to show that Ξ depends on $b_{s/2}^{(0)}(\alpha)$ and $b_{s/2}^{(1)}(\alpha)$ only for some s . Furthermore, if $|q_1| + \dots + |q_8| = r$, then it can be shown that Ξ depends on a unique value of s , given by (Laskar and Robutel, 1995, Eq. (75))

$$s = 1 + r + (r \bmod 2). \quad (12)$$

Looking back at Eq. (1), the coefficient Ξ has contributions from the term in $1/|\mathbf{r}_i - \mathbf{r}_j|$, usually called direct contributions, and contributions from the term in $\tilde{\mathbf{r}}_i \cdot \tilde{\mathbf{r}}_j$, usually called indirect contributions. The methods *direct*([k_1, k_2]) and *indirect*([k_1, k_2]) from class *PerHam* of the python package *celerics* give the full coefficient in front of $e^{\iota(k_1\lambda_i + k_2\lambda_j)}$ in Eq. (7). The method *direct* leaves out a factor $-\mathcal{G}m_i m_j / a_j = -m_i n_j \Lambda_j / m_0$. Disregarding ε in front of 1, the method *indirect* leaves out a factor $\alpha_{ij}^{-1/2} m_i n_j \Lambda_j / m_0$.

As an example, the term of $\varepsilon \mathcal{H}_{i,j}$ proportional to $e^{\iota(\lambda_i - 2\lambda_j)}$ at degree 1 in eccentricity/inclination is

$$\begin{aligned} & \frac{m_i}{m_0} n_j \Lambda_j (\Xi_i X_i + \Xi_j X_j), \\ \Xi_i &= -\frac{1}{3} \alpha_{ij}^{-1} b_{3/2}^{(1)}(\alpha_{ij}) + \frac{1}{2} b_{3/2}^{(0)}(\alpha_{ij}) - \frac{7}{12} \alpha_{ij} b_{3/2}^{(1)}(\alpha_{ij}) + \frac{5}{4} \alpha_{ij}^2 b_{3/2}^{(0)}(\alpha_{ij}) - \frac{5}{6} \alpha_{ij}^3 b_{3/2}^{(1)}(\alpha_{ij}), \quad (13) \\ \Xi_j &= \frac{1}{2} b_{3/2}^{(1)}(\alpha_{ij}) - \frac{5}{4} \alpha_{ij} b_{3/2}^{(0)}(\alpha_{ij}) + \frac{3}{4} \alpha_{ij}^2 b_{3/2}^{(1)}(\alpha_{ij}) + \frac{1}{2} \alpha_{ij}^{-1/2}. \end{aligned}$$

Of course, the Hamiltonian is real, so the term of $\varepsilon \mathcal{H}_P$ proportional to $e^{\iota(-\lambda_i + 2\lambda_j)}$ is the complex conjugated of Eq. (13). At degree 1, these are the only terms in front of $e^{\iota(\lambda_i - 2\lambda_j)}$. In particular, there is no dependency in Y_i and Y_j , as can be deduced from the d'Alembert rule and its extension.

In Table 1, I show that Eq. (7) can be reduced to a finite number of terms even when fast angles are not averaged, using the fact that Ξ goes to 0 when (k_1, k_2) goes to infinity.

p	1	2	5	20	50
Ξ_i	0.81133	0.69751	0.33960	0.0030273	0.000000715
$-\Xi_j$	0.44033	0.93170	0.47044	0.0042894	0.000000954

Table 1 — Coefficients Ξ_i and Ξ_j , respectively in front of $X_i e^{\iota(p\lambda_i - (p+1)\lambda_j)}$ and $X_j e^{\iota(p\lambda_i - (p+1)\lambda_j)}$, evaluated at $\alpha_{ij} = 0.7$, for some values of p . Both coefficients go to 0 when p goes to infinity.

¹It is assumed that i is the inner planet of the pair and j is the outer one. Hence $0 < \alpha_{ij} < 1$.

2 Expansion to second order in mass

The Hamiltonian is $\mathcal{H} = \mathcal{H}_K + \varepsilon \mathcal{H}_P$. In particular, it does not contain terms of order ε^2 or more without further approximations. Terms of order 2 in ε appear when averaging $\varepsilon \mathcal{H}_P$ over fast circulating angles. When the angle $k_1 \lambda_i + k_2 \lambda_j$ is fast-circulating (when the pair (i, j) is far from resonance $k_1 : -k_2$), then the term $e^{\iota(k_1 \lambda_i + k_2 \lambda_j)}$ in $\varepsilon \mathcal{H}_{i,j}$ averages to 0 and can be removed from $\varepsilon \mathcal{H}_P$ without affecting the dynamics too much. This procedure is justified mathematically using perturbation theory, and yields terms of order ε^2 in the Hamiltonian.

2.1 Perturbation theory : Lie serie expansion

The Hamiltonian of the planetary system being studied is

$$\mathcal{H}(\Lambda, \mathbf{x}, \mathbf{y}; \boldsymbol{\lambda}, -\iota \bar{\mathbf{x}}, -\iota \bar{\mathbf{y}}) = \mathcal{H}_K(\Lambda) + \varepsilon \mathcal{H}_P(\Lambda, \mathbf{x}, \mathbf{y}; \boldsymbol{\lambda}, -\iota \bar{\mathbf{x}}, -\iota \bar{\mathbf{y}}). \quad (14)$$

Since $\varepsilon \ll 1$, it is quasi-integrable, and if ε were to be neglected, its trajectories would simply be

$$\begin{aligned} \dot{\mathbf{x}} &= \iota \frac{\partial \mathcal{H}_K}{\partial \bar{\mathbf{x}}} = 0 \Rightarrow \mathbf{x}(t) = \mathbf{x}_0 \Rightarrow \mathbf{e}(t) = \mathbf{e}_0 \text{ and } \boldsymbol{\varpi}(t) = \boldsymbol{\varpi}_0, \\ \dot{\mathbf{y}} &= \iota \frac{\partial \mathcal{H}_K}{\partial \bar{\mathbf{y}}} = 0 \Rightarrow \mathbf{y}(t) = \mathbf{y}_0 \Rightarrow \mathbf{I}(t) = \mathbf{I}_0 \text{ and } \boldsymbol{\Omega}(t) = \boldsymbol{\Omega}_0, \\ \dot{\Lambda} &= \frac{\partial \mathcal{H}_K}{\partial \Lambda} = 0 \Rightarrow \Lambda(t) = \Lambda_0 \Rightarrow \mathbf{a}(t) = \mathbf{a}_0, \\ \dot{\boldsymbol{\lambda}} &= \frac{\partial \mathcal{H}_K}{\partial \Lambda} := \mathbf{n} \Rightarrow \boldsymbol{\lambda}(t) = \boldsymbol{\lambda}_0 + \mathbf{n}t. \end{aligned} \quad (15)$$

Disregarding ε is too rough of an approximation because it is equivalent to disregarding planet–planet interactions. In order to do better than that, I will look for a change of variable

$$\begin{aligned} \Psi : \mathbb{R}^{6n} &\rightarrow \mathbb{R}^{6n} \\ (\Lambda, \mathbf{x}, \mathbf{y}; \boldsymbol{\lambda}, -\iota \bar{\mathbf{x}}, -\iota \bar{\mathbf{y}}) &\mapsto (\Lambda', \mathbf{x}', \mathbf{y}'; \boldsymbol{\lambda}', -\iota \bar{\mathbf{x}}', -\iota \bar{\mathbf{y}}'), \end{aligned} \quad (16)$$

close to the identity, such that

$$\begin{aligned} \check{\mathcal{H}}(\Lambda', \mathbf{x}', \mathbf{y}'; \boldsymbol{\lambda}', -\iota \bar{\mathbf{x}}', -\iota \bar{\mathbf{y}}') &= \mathcal{H}(\Lambda, \mathbf{x}, \mathbf{y}; \boldsymbol{\lambda}, -\iota \bar{\mathbf{x}}, -\iota \bar{\mathbf{y}}) \\ &= \check{\mathcal{H}}_0(\Lambda') + \varepsilon \check{\mathcal{H}}_1 + \varepsilon^2 \check{\mathcal{H}}_2(\Lambda', \mathbf{x}', \mathbf{y}'; \boldsymbol{\lambda}', -\iota \bar{\mathbf{x}}', -\iota \bar{\mathbf{y}}'), \end{aligned} \quad (17)$$

where $\check{\mathcal{H}} = \mathcal{H} \circ \Psi^{-1}$ and $\check{\mathcal{H}}_1 = \langle \mathcal{H}_P \rangle$. The operator $\langle \cdot \rangle$ denotes the average over the fast combinations of 2-planet angles $k_1 \lambda_i + k_2 \lambda_j$ in Eq. (7). That is to say, $\check{\mathcal{H}}_1$ depends only on slow combinations of angles. In order to ensure a canonical transformation, I choose for Ψ the flow at time -1 of a generator χ that I will constrain in such a way that only slow angles appear in $\varepsilon \check{\mathcal{H}}_1$.

$$\Psi = \Phi_{\varepsilon \chi}(-1, \cdot), \quad \text{or equivalently} \quad \Psi^{-1} = \Phi_{\varepsilon \chi}(1, \cdot). \quad (18)$$

For more details about flows, generators and perturbation theory, refer to Couturier, 2022, Sects. 2.1.3 & 2.2.2, and reference therein. I define the Lie derivative of χ as the time-derivative along the trajectories of χ , that is

$$L_\chi = \{\chi, \cdot\} = \frac{d}{dt}\Big|_\chi = \frac{\partial \chi}{\partial \Lambda'} \cdot \frac{\partial}{\partial \Lambda'} - \frac{\partial \chi}{\partial \Lambda'} \cdot \frac{\partial}{\partial \Lambda'} + \iota \frac{\partial \chi}{\partial \mathbf{x}'} \cdot \frac{\partial}{\partial \bar{\mathbf{x}}'} - \iota \frac{\partial \chi}{\partial \bar{\mathbf{x}}'} \cdot \frac{\partial}{\partial \mathbf{x}'} + \iota \frac{\partial \chi}{\partial \mathbf{y}'} \cdot \frac{\partial}{\partial \bar{\mathbf{y}}'} - \iota \frac{\partial \chi}{\partial \bar{\mathbf{y}}'} \cdot \frac{\partial}{\partial \mathbf{y}}. \quad (19)$$

I can now give $\check{\mathcal{H}}$ as the Taylor expansion

$$\check{\mathcal{H}} = \mathcal{H} \circ \Psi^{-1} = \mathcal{H} \circ \Phi_{\varepsilon\chi}(1, \cdot) = \sum_{j=0}^{+\infty} \frac{L_{\varepsilon\chi}^{(j)}}{j!}(\mathcal{H}) = e^{\varepsilon L_\chi}(\mathcal{H}), \quad (20)$$

where $L_{\varepsilon\chi}^{(j)}$ denotes the j^{th} iteration of the Lie derivative, or the j^{th} time-derivative along the flow of $\varepsilon\chi$. If I expand Eq. (20) to second order in ε , I obtain

$$\check{\mathcal{H}}_0 + \varepsilon \check{\mathcal{H}}_1 + \varepsilon^2 \check{\mathcal{H}}_2 = \mathcal{H}_K + \varepsilon \left(\mathcal{H}_P + \{\chi, \mathcal{H}_K\} \right) + \varepsilon^2 \left(\{\chi, \mathcal{H}_P\} + \frac{1}{2} \{\chi, \{\chi, \mathcal{H}_K\}\} \right), \quad (21)$$

and in order to achieve the form (17), I constrain the generator of the transformation with the cohomological equation

$$\mathcal{H}_P - \langle \mathcal{H}_P \rangle = \{\mathcal{H}_K, \chi\}. \quad (22)$$

The cohomological Eq. (22) now constrains the generator χ . Injecting it into Eq. (21), the expression of the Hamiltonian at second order in mass is

$$\check{\mathcal{H}}_2 = \frac{1}{2} \{\chi, \mathcal{H}_P + \langle \mathcal{H}_P \rangle\}. \quad (23)$$

The Hamiltonian at second order in mass is entirely determined by Eq. (23) provided that the generator χ can be extracted from Eq. (22). To do so, I write (see Eq. (6))

$$\chi = \sum_{i=1}^n \sum_{j=i+1}^n \chi_{i,j}, \quad \mathcal{H}_P = \sum_{i=1}^n \sum_{j=i+1}^n \mathcal{H}_{i,j}, \quad (24)$$

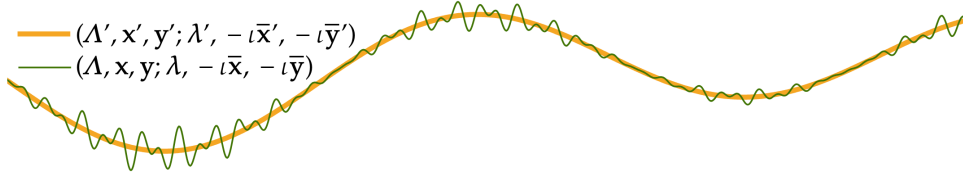
and then in the Fourier domain

$$\chi_{i,j} = \sum_{\mathbf{k} \in \mathbb{Z}^2 \setminus (0,0)} \chi_{\mathbf{k}} e^{\iota(k_1 \lambda_i + k_2 \lambda_j)}, \quad \mathcal{H}_{i,j} - \langle \mathcal{H}_{i,j} \rangle = \sum_{\mathbf{k} \in \mathbb{Z}^2 \setminus (0,0)} h_{\mathbf{k}} e^{\iota(k_1 \lambda_i + k_2 \lambda_j)}. \quad (25)$$

Injecting into Eq. (22) and using $\partial \mathcal{H}_K / \partial \Lambda = \mathbf{n}$ yields

$$\chi_{\mathbf{k}} = \frac{-\iota h_{\mathbf{k}}}{k_1 n_i + k_2 n_j}. \quad (26)$$

When the 2-planet angle $k_1 \lambda_i + k_2 \lambda_j$ is among the combinations averaged by $\langle \cdot \rangle$, the corresponding coefficient $h_{\mathbf{k}}$ is non-zero in Eq. (26). If furthermore the averaged combination is a slow angle, then the denominator can be very small, and the transformation Ψ can be very far from identity. However, as long as the operator $\langle \cdot \rangle$ averages only over fast

Fig. 1 — Schema of the transformation Ψ

angles, then the denominator of Eq. (26) is of order unity when h_k is non-zero, and Ψ is a quasi-periodic transformation ε -close from identity.

In other words, the variables $(\Lambda', \mathbf{x}', \mathbf{y}'; \lambda', -\iota \bar{\mathbf{x}}', -\iota \bar{\mathbf{y}}')$ differ from the original non-primed variables by a quasi-periodic transformation ε -close from identity. These new variables contain the tendency of the trajectories, where the short-period variations were removed (see the schema of Fig. 1), and are more useful for an analytical work than the original variables. From now on, I will be working with $\check{\mathcal{H}}$ and the primed variables only and I will omit the prime and the $\checkmark$ for clarity.

2.2 Expansion in Poincaré coordinates and d'Alembert rule

Analyzing Eqs. (23) and (24), it can be noticed that at second order in mass, the Hamiltonian does depend on 3-planet angles. For example, considering the pair $(i, j) = (1, 2)$ on the left-side of the Poisson bracket in Eq. (23), and the pair $(i, j) = (1, 3)$ on the right-side yields terms of the form $e^{\iota(k_1\lambda_1+k_2\lambda_2+k_3\lambda_3)}$. Likewise, dependency on 2-planet, 1-planet and 0-planet angles remain in $\mathcal{H}_2$. Clearly, $\mathcal{H}_2$ has no dependency on 5-planet angles or more. Dependency on 4-planet angles should come from brackets of the form $\{e^{\iota(p\lambda_i+q\lambda_j)}, e^{\iota(r\lambda_k+s\lambda_l)}\}$ where (i, j, k, l) are all different, but then the sets of variables on the left-side and right-side are disjoint and the bracket vanishes due to Eq. (19). Therefore, $\mathcal{H}_2$ has no dependency on 4-planet angles and can be written

$$\varepsilon^2 \mathcal{H}_2 = \sum_{i=1}^n \sum_{j=i+1}^n \sum_{k=j+1}^n \varepsilon^2 \mathcal{H}_{i,j,k}, \quad (27)$$

which emphasizes the fact that it only contains interactions between three planets. Using Eq. (7), I have²

$$\mathcal{H}_{i,j,k} = \sum_{\mathbf{k} \in \mathbb{Z}^3} \left(\sum_{\mathbf{q} \in \mathbb{N}^{12}} \Xi X_i^{q_1} X_j^{q_2} X_k^{q_3} \bar{X}_i^{q_4} \bar{X}_j^{q_5} \bar{X}_k^{q_6} Y_i^{q_7} Y_j^{q_8} Y_k^{q_9} \bar{Y}_i^{q_{10}} \bar{Y}_j^{q_{11}} \bar{Y}_k^{q_{12}} \right) e^{\iota(k_1\lambda_i+k_2\lambda_j+k_3\lambda_k)}. \quad (28)$$

The considerations of symmetry stated in Sect. 1.3 remain valid at any order in mass, and the d'Alembert rule and its extension read

$$\begin{aligned} k_1 + k_2 + k_3 + q_1 + q_2 + q_3 - q_4 - q_5 - q_6 + q_7 + q_8 + q_9 - q_{10} - q_{11} - q_{12} &= 0, \\ q_7 + q_8 + q_9 + q_{10} + q_{11} + q_{12} &= 0 \pmod{2}. \end{aligned} \quad (29)$$

² $\mathbf{k} = (k_1, k_2, k_3)$ is the 3-planet angle and k is the index of the outermost planet of the triplet.

2.3 Practical considerations

From now on I will assume, without restriction to the generality and to lighten the notations, that the 3-planet angle of interest is $k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3$. In order to obtain the term of $\varepsilon^2\mathcal{H}_2$ proportional to $e^{\iota(k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3)}$ at order r in eccentricity and inclination, all Poisson brackets of the form

$$\frac{1}{2} \left\{ \frac{-\iota h_{\mathbf{p}}}{p_1 n_i + p_2 n_j} e^{\iota(p_1 \lambda_i + p_2 \lambda_j)}, h_{\mathbf{p}'} e^{\iota(p'_1 \lambda_{i'} + p'_2 \lambda_{j'})} \right\}, \quad (30)$$

such that $p_1\lambda_i + p_2\lambda_j + p'_1\lambda_{i'} + p'_2\lambda_{j'} = k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3$, for $(i, j, i', j') \in \{1, 2, 3\}^4$, have to be considered in Eq. (23). If $k_1 k_2 k_3 \neq 0$, then the 4-tuple (i, j, i', j') must belong to the set $\{(1, 2, 1, 3), (1, 3, 1, 2), (1, 3, 2, 3), (2, 3, 1, 3), (1, 2, 2, 3), (2, 3, 1, 2)\}$. If $k_1 k_2 k_3 = 0$, then there are more possibilities for the value of (i, j, i', j') . In fact, that tuple does not even have to belong to $\{1, 2, 3\}^4$ in that case. Indeed, if $k_3 = 0$, even brackets of the form $\{e^{\iota(p_1\lambda_1 + p_2\lambda_5)}, e^{\iota(p'_1\lambda_2 + p'_2\lambda_5)}\}$ can generate contributions of the form $e^{\iota(k_1\lambda_1 + k_2\lambda_2)}$. However, when $k_1 k_2 k_3 = 0$, the angle $k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3$ also exists at order 1 in ε and its contribution of second order is often not important. Therefore, I will restrict myself to the case $k_1 k_2 k_3 \neq 0$ in this work.

The Poisson bracket in Eq. (30) must be of degree r or less in eccentricity and inclination. If $h_{\mathbf{p}}$ is of degree s and $h_{\mathbf{p}'}$ is of degree s' , then the full bracket is of degree $s + s'$ if $ss' = 0$ and of degree $s + s' - 2$ if $ss' \neq 0$ (by differentiating with respect to the eccentricities and inclinations in Eq. (19)). The largest possible size for both s and s' is therefore $r + 1$ and $\varepsilon\mathcal{H}_P$ needs to be expanded at degree $r + 1$ to get $\varepsilon^2\mathcal{H}_2$ at degree r .

For the 3-planet angle $k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3$, there are infinitely many brackets to consider if $k_1 k_2 k_3 = 0$. Even if a truncation in the size of Ξ can be set (see Table 1), far too many Poisson brackets must be computed and the final analytical expression is ridiculously untractable. However, when $k_1 k_2 k_3 \neq 0$, it is straightforward to notice that there are only finitely many brackets to consider and to give them explicitly. This gives another argument in favor of restricting oneself to $k_1 k_2 k_3 \neq 0$. Consider for instance the case $(i, j, i', j') = (1, 2, 1, 3)$. The reasoning is essentially the same for the five other possible values of that tuple. Then the condition $p_1\lambda_i + p_2\lambda_j + p'_1\lambda_{i'} + p'_2\lambda_{j'} = k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3$ yields

$$\begin{aligned} p_1 + p'_1 &= k_1, \\ p_2 &= k_2, \\ p'_2 &= k_3. \end{aligned} \quad (31)$$

The condition that the degree of $h_{\mathbf{p}}$ and $h_{\mathbf{p}'}$ must be less than $r + 1$ furthermore adds the bounds $|p_1 + p_2| \leq r + 1$ and $|p'_1 + p'_2| \leq r + 1$ because of the d'Alembert rule (see Eq. (8)). Since p_2 and p'_2 are forced to particular values, the equation $p_1 + p'_1 = k_1$ has only finitely many solutions that satisfy these two additional bounds. As an example, for $(k_1, k_2, k_3) = (2, -4, 3)$ at degree 1, assuming that all 2-planet angles are fast when defining $\langle \cdot \rangle$, there are a total of 24 Poisson brackets of the form (30) to compute, four for each value of the tuple $(i, j, i', j') \in \{(1, 2, 1, 3), (1, 3, 1, 2), (1, 3, 2, 3), (2, 3, 1, 3), (1, 2, 2, 3), (2, 3, 1, 2)\}$.

2.4 Expression as a function of the Laplace coefficients

The method *Angle* of the module *PerHam3pla* of the python package *celerier* gives the full coefficient in front of $e^{i(k_1\lambda_1+k_2\lambda_2+k_3\lambda_3)}$ in $\varepsilon^2\mathcal{H}_2$ at any degree in eccentricity and inclination, when $k_1k_2k_3 \neq 0$. It leaves out a factor $m_1m_2n_3\Lambda_3/m_0^2$. The output of this function depends on the Laplace coefficients $b_{s/2}^{(k)}(\alpha_{12})$, $b_{s/2}^{(k)}(\alpha_{13})$ and $b_{s/2}^{(k)}(\alpha_{23})$ with $k \in \{0, 1\}$ in virtue of Eq. (11). Because the Laplace coefficients depend on Λ , Ξ should depend on $db_{s/2}^{(k)}/d\alpha$ as well. However, the relation

$$\frac{db_{s/2}^{(k)}}{d\alpha} = \frac{s^2}{s-2k}b_{(s+2)/2}^{(k-1)}(\alpha) + \left(\frac{s^2-ks}{2k-s}\alpha + \frac{ks}{2k-s}\alpha^{-1}\right)b_{(s+2)/2}^{(k)}(\alpha) \quad (32)$$

is used to remove dependency on $db_{s/2}^{(k)}/d\alpha$. From Eq. (12), the maximum value of s in $\varepsilon\mathcal{H}_P$ at degree r in eccentricity and inclination is $s = 1 + r + (r \bmod 2)$. Using Eq. (32), the maximum value of s in $\varepsilon^2\mathcal{H}_2$ is³

$$s = 3 + r + (r \bmod 2). \quad (33)$$

The module *PerHam3pla* takes the following set of parameters as argument:

- **degree.** The degree of expansion in eccentricity and inclination. No default value.
- **ang2pla** = $[(i_1, j_1, p_1, q_1), \dots, (i_m, j_m, p_m, q_m)]$. A set of m 2-planet angles that should not be averaged when defining the operator $\langle \cdot \rangle$. Angles $p_1\lambda_{i_1} + q_1\lambda_{j_1}$ to $p_m\lambda_{i_m} + q_m\lambda_{j_m}$ will not be averaged. $(i_k, j_k) \in \{(1, 2), (1, 3), (2, 3)\}$. Only the fundamental of a 2-planet angle or its opposite must be mentioned. Harmonics must not be mentioned ($\gcd(p_k, q_k) = 1$). Defaults to $[]$.
- **n0** = (n_1, n_2, n_3) . The function can give its output evaluated at mean motions given by that parameter. Units do not matter as only the ratios are needed. Semi-major axes are obtained from $a_i/a_j = (n_j/n_i)^{2/3}$ and do not need to be specified. That parameter is only important if an evaluation is required and can be left to its default value $()$ otherwise.
- **ev.** A boolean deciding if the output should be evaluated at the nominal mean motions provided. Defaults to False.
- **spatial.** A boolean deciding if the problem is planar (False) or 3D (True). The output does not depend on Y_i or $\bar{Y}_i$ when False. Defaults to False.
- **keplerian.** A boolean deciding whether or not the equality $k_1n_1 + k_2n_2 + k_3n_3 = 0$ should be assumed True and used to simplify the output. Defaults to False.
- **disregard13.** A boolean parameter controlling if contributions from the pair (1, 3) are disregarded. Defaults to False.
- **disregardInd.** A boolean parameter controlling if contributions from indirect terms in $\varepsilon\mathcal{H}_P$ (coming from $\tilde{\mathbf{r}}_i \cdot \tilde{\mathbf{r}}_j$) are disregarded. Defaults to False.

³No derivation with respect to Λ occurs when a term of the bracket in Eq. (30) is of degree $r + 1$.

- **takeout_kn**. In Eq. (30), the denominator $(p_1 n_i + p_2 n_j)^{-1}$ depends on Λ through $\partial n_j / \partial \Lambda_j = -3n_j / \Lambda_j$ and cannot be taken out of the Poisson bracket a priori as it disregards terms of order ε^2 . However, that denominator can still be factored out of the Poisson bracket by setting this boolean to True. Defaults to False.
- **verbose**. A boolean determining if the method *Angle* prints what Poisson brackets it computes. If False, the method is mute. Defaults to True.

The method *Angle* takes as single argument the tuple $= (k_1, k_2, k_3)$.

2.5 Application to K2-138

K2-138 is a planetary system 661 ± 7 light-years away with a $0.93M_\odot$ star orbited by six planets whose periods are given in Table 2. The masses are very poorly constrained but are a few Earth masses for each planet.

planet	b	c	d	e	f	g
P_j	1	1.513	2.297	3.511	5.422	17.835
$2\pi/P_j$	2π	4.1529	2.7353	1.78955	1.15886	0.3523
n_j^*	2π	4.1539	$(5n_c^* - 2n_b^*)/3$	$(5n_d^* - 2n_c^*)/3$	$(5n_e^* - 2n_d^*)/3$	$(4n_f^* - 2n_e^*)/3$

Table 2 — The K2-138 system. Periods are normalized by $P_b = 2.353$ days. The nominal values n_j^* are as close as possible from n_j such that the derivatives of all Laplace angles are zero.

Triplets (b,c,d), (c,d,e) and (d,e,f) are close from the 3-planet resonance $2n_1 - 5n_2 + 3n_3 = 0$. This is a resonance of degree 0 in eccentricity and inclination. The triplet (e,f,g) is close from the 3-planet resonance $2n_1 - 4n_2 + 3n_3 = 0$ which is of degree 1 in eccentricity and inclination.

2.5.1 Triplet (b,c,d)

While the pairs (b,c) and (c,d) are already rather far from resonance $2 : 3$, the triplet (b,c,d) is close from a 0th degree 3-planet resonance since $2n_b - 5n_c + 3n_d \approx 0.0078$. Denoting (b,c,d) = (1, 2, 3), the 3-planet Laplace angle of interest is therefore $\phi = 2\lambda_1 - 5\lambda_2 + 3\lambda_3$. According to the 3-planet d'Alembert rule (Eq. (29)), the term of $\mathcal{H}_2$ proportional to $e^{i(2\lambda_1 - 5\lambda_2 + 3\lambda_3)}$ at degree 2 in eccentricity and inclination writes

$$\varepsilon^2 \mathcal{H}_2 = \frac{1}{2} \frac{m_1}{m_0} \frac{m_2}{m_0} n_3 \Lambda_3 \left(\Xi + \sum_{1 \leq i, j \leq 3} \Xi_{ij} X_i \bar{X}_j + \sum_{1 \leq i, j \leq 3} \Xi'_{ij} Y_i \bar{Y}_j \right) e^{i\phi}. \quad (34)$$

In particular, the eccentric and inclined dynamics are decoupled because of the extension to the d'Alembert rule. Since all planets of K2-138 are transiting, I will disregard the inclined dynamics altogether from now on. Because this 3-planet resonance exists at degree 0, it is interesting to first understand the circular dynamics. At zeroth degree in eccentricity, the harmonics of the resonant angle also contribute. Combining with the complex conjugate to get a real Hamiltonian, I obtain

$$\varepsilon^2 \mathcal{H}_2 = \frac{m_1}{m_0} \frac{m_2}{m_0} n_3 \Lambda_3 \sum_{j \in \mathbb{N}^*} \Xi_j \cos j\phi \quad (35)$$

Evaluated at the nominal mean motions of Table 2, the coefficients Ξ_j quickly go to zero as j goes to infinity. Making no approximation (pair (1, 3) and indirect terms are considered and denominator is kept inside the Poisson bracket for derivation), I computed these coefficients with the module *PerHam3pla* in python. In the first computation, I defined the operator $\langle \cdot \rangle$ by averaging over all 2-planet angles, while in the second computation, the 2-planet angles $2\lambda_1 - 3\lambda_2$ and $2\lambda_2 - 3\lambda_3$ were considered slow and not averaged upon. In that second case, terms of $\varepsilon\mathcal{H}_P$ depending on those 2-planets angles must be kept in the Hamiltonian. In Table 3, the results of these computations are presented.

j	1	2	3	4	5	6	10
$-\Xi_j$	180.8599	0.74794	0.08100	0.01237	0.002187	0.0004195	0.000000831
$-\Xi'_j$	-7.407405	0.74794	0.08100	0.01237	0.002187	0.0004195	0.000000831

Table 3 — Coefficients Ξ_j appearing in Eq. (35) evaluated at the nominal mean motions n_j^* of Table 2. From $j = 2$, the coefficients are very small and both computations coincide as the denominators $2n_1 - 3n_2$ and $2n_2 - 3n_3$ never appear anymore in Eq. (30).

Since Ξ_2 is negligible with respect to Ξ_1 , I will now limit myself to

$$\varepsilon^2 \mathcal{H}_2 = \frac{m_1}{m_0} \frac{m_2}{m_0} n_3 \Lambda_3 \Xi \cos \phi. \quad (36)$$

The module *PerHam3pla* allows three approximations:

- All Poisson brackets involving the pair (1, 3) are disregarded (`disregard13` is `True`),
- All indirect terms (coming from $\tilde{\mathbf{r}}_i \cdot \tilde{\mathbf{r}}_j$) are disregarded (`disregardInd` is `True`),
- The denominator $(p_1 n_i + p_2 n_j)^{-1}$ is factored out (`takeout_kn` is `True`).

In order to evaluate them, I computed Ξ in Eq. (36) with all eight possible combinations of these approximations (`True` or `False` for each). In Table 4, approximation 100 means that all Poisson brackets involving the pair (1, 3) are disregarded, but indirect terms are considered and the denominator $(p_1 n_i + p_2 n_j)^{-1}$ is not factored out of the Poisson bracket.

In the case where $\langle \cdot \rangle$ is defined by averaging over all 2-planet angles, and without making any approximation (**`disregard13`** = **`disregardInd`** = **`takeout_kn`** = `False`), I provide [here](https://jeremycouturier.com/3pla/253.txt)⁴ the analytical expression of Ξ in Eq. (36) as a function of the Laplace coefficients.

2.5.2 Triplet (e,f,g)

While the pairs (e,f) and (f,g) are already rather far from resonance 2 : 3 and 1 : 3, this triplet is very close from a 1st degree 3-planet resonance since $2n_e - 4n_f + 3n_g \approx 0.00056$. Denoting (e,f,g) = (1, 2, 3), the 3-planet Laplace angle of interest is therefore

⁴<https://jeremycouturier.com/3pla/253.txt>

Appx.	000	001	010	011	100	101	110	111
$-\Xi$	180.860	182.604	180.860	182.604	181.627	183.639	181.627	183.639
Ξ'	7.40740	5.66305	7.40740	5.66305	6.63983	4.62852	6.63983	4.62852

Table 4 — Coefficients Ξ appearing in Eq. (36) for all possible approximations evaluated at the nominal mean motions n_j^* of Table 2. The line with Ξ' corresponds to the case where $2\lambda_1 - 3\lambda_2$ and $2\lambda_2 - 3\lambda_3$ are not averaged upon. For that 3-planet angle and that degree of expansion, the indirect terms approximation has no influence, but that is not always the case.

$\phi = 2\lambda_1 - 4\lambda_2 + 3\lambda_3$. According to the 3-planet d'Alembert rule (Eq. (29)), the term of $\mathcal{H}_2$ proportional to $e^{i(2\lambda_1 - 4\lambda_2 + 3\lambda_3)}$ at degree 1 in eccentricity and inclination reads

$$\varepsilon^2 \mathcal{H}_2 = \frac{m_1 m_2}{m_0^2} n_3 \Lambda_3 \left(\Xi_1 \bar{X}_1 + \Xi_2 \bar{X}_2 + \Xi_3 \bar{X}_3 \right) e^{i\phi}. \quad (37)$$

In particular, the expansion to the d'Alembert rule shows that the problem is planar at degree 1 in eccentricity and inclination. There are no terms of degree 0 and the harmonics of ϕ play no role at lowest degree. Like for the triplet (b,c,d), I evaluate the quality of the eight possible combinations of approximations by computing Ξ_1 , Ξ_2 and Ξ_3 in all cases. Results are presented in Table 5.

In the case where $\langle \cdot \rangle$ is defined by averaging over all 2-planet angles, and without making any approximation (**disregard13** = **disregardInd** = **takeout_kn** = False), I provide [here](https://jeremycouturier.com/3pla/243.txt)⁵ the analytical expressions of Ξ_1 , Ξ_2 and Ξ_3 in Eq. (37) as a function of the Laplace coefficients.

Appx.	000	001	010	011	100	101	110	111
$-\Xi_1$	0.231	0.2471	0.2661	0.3189	-0.076	-0.112	-0.042	-0.040
Ξ_2	10.859	11.279	10.859	11.279	10.874	11.304	10.874	11.304
$-\Xi_3$	20.548	21.258	20.548	21.258	21.395	22.283	21.395	22.283
$-\Xi'_1$	0.231	0.2471	0.2661	0.3189	-0.076	-0.112	-0.042	-0.040
$-\Xi'_2$	1.4925	1.0727	1.4925	1.0727	1.4770	1.0476	1.4770	1.0476
Ξ'_3	3.7489	3.0392	3.7489	3.0392	2.9017	2.0142	2.9017	2.0142

Table 5 — Coefficients Ξ_j appearing in Eq. (37) for all possible approximations evaluated at the nominal mean motions n_j^* of Table 2. The lines with Ξ'_j corresponds to the case where $2\lambda_1 - 3\lambda_2$ and $\lambda_2 - 3\lambda_3$ are not averaged upon.

2.6 The 3-planet angle $p\lambda_1 - (p+q)\lambda_2 + q\lambda_3$

In this subsection, I consider 3-planet angles $k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3$ where $k_1 + k_2 + k_3 = 0$, and $k_1 k_2 k_3 \neq 0$. For these angles, there exist non-zero integers (p, q) such that $k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = p\lambda_1 - (p+q)\lambda_2 + q\lambda_3$. That is for example the case of the triplet (b,c,d) of K2-138 in Sect. 2.5.1 with $p = 2$ and $q = 3$. For these 3-planet angles, the d'Alembert

⁵<https://jeremycouturier.com/3pla/243.txt>

rule (Eq. (29)) shows that terms of degree zero in eccentricity and inclination exist. I will compute by hand the coefficient Ξ such that the Hamiltonian to second order in mass and to degree zero reads

$$\varepsilon^2 \mathcal{H}_2 = \frac{m_1 m_2}{m_0^2} n_3 \Lambda_3 \Xi \cos(p\lambda_1 - (p+q)\lambda_2 + q\lambda_3). \quad (38)$$

Denoting $\phi = p\lambda_1 - (p+q)\lambda_2 + q\lambda_3$, the term $\cos \phi$ in Eq. (38) comes from $\Xi/2 (e^{i\phi} + e^{-i\phi})$, so I will simply obtain and double the term in front of $e^{i\phi}$. For the calculations to be doable by hand, I disregard contributions from pair (1, 3) and from indirect terms. However, I do not factor the denominator $(p_1 n_i + p_2 n_j)^{-1}$ out of the Poisson brackets. I define the operator $\langle \cdot \rangle$ (see Eq. (17)) by averaging over all 2-planet angles. There exist a total of six brackets of the form of Eq. (30) that give a contribution in $e^{i\phi}$. These are

$$\begin{aligned} & \frac{-\iota}{2} \left\{ \frac{m_1}{m_0} n_2 \Lambda_2 \frac{W_{12}^{(p)}}{p(n_1 - n_2)} e^{i p(\lambda_1 - \lambda_2)}, \frac{m_2}{m_0} n_3 \Lambda_3 W_{23}^{(q)} e^{-i q(\lambda_2 - \lambda_3)} \right\}, \\ & \frac{\iota}{2} \left\{ \frac{m_2}{m_0} n_3 \Lambda_3 \frac{W_{23}^{(q)}}{q(n_2 - n_3)} e^{-i q(\lambda_2 - \lambda_3)}, \frac{m_1}{m_0} n_2 \Lambda_2 W_{12}^{(p)} e^{i p(\lambda_1 - \lambda_2)} \right\}, \end{aligned} \quad (39)$$

$$\begin{aligned} & \frac{-\iota}{2} \left\{ \frac{m_1}{m_0} n_2 \Lambda_2 \frac{V_{12,>}^{(p)}}{p n_1 - (p+1)n_2} x_2 e^{i(p\lambda_1 - (p+1)\lambda_2)}, \frac{m_2}{m_0} n_3 \Lambda_3 V_{23,<}^{(q-1)} \bar{x}_2 e^{-i((q-1)\lambda_2 - q\lambda_3)} \right\}, \\ & \frac{\iota}{2} \left\{ \frac{m_2}{m_0} n_3 \Lambda_3 \frac{V_{23,<}^{(q-1)}}{(q-1)n_2 - q n_3} \bar{x}_2 e^{-i((q-1)\lambda_2 - q\lambda_3)}, \frac{m_1}{m_0} n_2 \Lambda_2 V_{12,>}^{(p)} x_2 e^{i(p\lambda_1 - (p+1)\lambda_2)} \right\}, \end{aligned} \quad (40)$$

$$\begin{aligned} & \frac{-\iota}{2} \left\{ \frac{m_1}{m_0} n_2 \Lambda_2 \frac{V_{12,>}^{(-p)}}{p n_1 - (p-1)n_2} \bar{x}_2 e^{i(p\lambda_1 - (p-1)\lambda_2)}, \frac{m_2}{m_0} n_3 \Lambda_3 V_{23,<}^{(-q-1)} x_2 e^{-i((q+1)\lambda_2 - q\lambda_3)} \right\}, \\ & \frac{\iota}{2} \left\{ \frac{m_2}{m_0} n_3 \Lambda_3 \frac{V_{23,<}^{(-q-1)}}{(q+1)n_2 - q n_3} x_2 e^{-i((q+1)\lambda_2 - q\lambda_3)}, \frac{m_1}{m_0} n_2 \Lambda_2 V_{12,>}^{(-p)} \bar{x}_2 e^{i(p\lambda_1 - (p-1)\lambda_2)} \right\}. \end{aligned} \quad (41)$$

For the two brackets (39), the operator $\{\cdot, \cdot\}$ reduces to $\frac{\partial}{\partial \Lambda_2} \frac{\partial}{\partial \lambda_2} - \frac{\partial}{\partial \lambda_2} \frac{\partial}{\partial \Lambda_2}$ whereas $\{\cdot, \cdot\} = \iota \frac{\partial}{\partial x_2} \frac{\partial}{\partial \bar{x}_2} - \iota \frac{\partial}{\partial \bar{x}_2} \frac{\partial}{\partial x_2}$ for the last four brackets. When indirect terms are disregarded, $\varepsilon \mathcal{H}_P$ is expanded with (Petit, 2021, Eqs. (12) to (15))

$$\begin{aligned} W_{ij}^{(l)} &= -\frac{1}{2} b_{1/2}^{(l)}(\alpha_{ij}), \\ V_{ij,<}^{(l)} &= \frac{1}{2} \sqrt{\frac{2}{\Lambda_i}} \left(l + 1 + \frac{\alpha_{ij}}{2} \frac{\partial}{\partial \alpha_{ij}} \right) b_{1/2}^{(l+1)}(\alpha_{ij}), \\ V_{ij,>}^{(l)} &= -\frac{1}{2} \sqrt{\frac{2}{\Lambda_j}} \left(l + \frac{1}{2} + \frac{\alpha_{ij}}{2} \frac{\partial}{\partial \alpha_{ij}} \right) b_{1/2}^{(l)}(\alpha_{ij}), \end{aligned} \quad (42)$$

and all six brackets are efficiently computed by hand using the expressions $\partial \alpha_{ij} / \partial \Lambda_i = 2\alpha_{ij} / \Lambda_i$, $\partial \alpha_{ij} / \partial \Lambda_j = -2\alpha_{ij} / \Lambda_j$ and $\partial n_j / \partial \Lambda_j = -3n_j / \Lambda_j$. A factor $m_1 m_2 n_3 \Lambda_3 / m_0^2$ can be immediately factored out of the first two brackets. For the last four brackets, nearly everything can be factored out and I am left with $\{x_2, \bar{x}_2\} = -\{\bar{x}_2, x_2\} = \iota$. In order to

give the final result as a function of n_2 and n_3 only, I assume $pn_1 - (p+q)n_2 + qn_3 = 0$. I obtain

$$\begin{aligned} \Xi = & \frac{n_2}{n_2 - n_3} \left(b_{1/2}^{(p)} b_{1/2}^{(q)} + \alpha_{12} b_{1/2}^{(q)} \frac{db_{1/2}^{(p)}}{d\alpha_{12}} + \frac{p}{q} \alpha_{23} b_{1/2}^{(p)} \frac{db_{1/2}^{(q)}}{d\alpha_{23}} \right) + \frac{3p}{2q} \frac{n_2^2}{(n_2 - n_3)^2} b_{1/2}^{(p)} b_{1/2}^{(q)} \\ & - \frac{n_2/2}{(q-1)n_2 - qn_3} \left(q(2p+1) b_{1/2}^{(p)} b_{1/2}^{(q)} + q\alpha_{12} b_{1/2}^{(q)} \frac{db_{1/2}^{(p)}}{d\alpha_{12}} + \left(\frac{1}{2} + p\right) \alpha_{23} b_{1/2}^{(p)} \frac{db_{1/2}^{(q)}}{d\alpha_{23}} + \frac{\alpha_{13}}{2} \frac{db_{1/2}^{(p)}}{d\alpha_{12}} \frac{db_{1/2}^{(q)}}{d\alpha_{23}} \right) \\ & + \frac{n_2/2}{(q+1)n_2 - qn_3} \left(q(2p-1) b_{1/2}^{(p)} b_{1/2}^{(q)} - q\alpha_{12} b_{1/2}^{(q)} \frac{db_{1/2}^{(p)}}{d\alpha_{12}} + \left(\frac{1}{2} - p\right) \alpha_{23} b_{1/2}^{(p)} \frac{db_{1/2}^{(q)}}{d\alpha_{23}} + \frac{\alpha_{13}}{2} \frac{db_{1/2}^{(p)}}{d\alpha_{12}} \frac{db_{1/2}^{(q)}}{d\alpha_{23}} \right), \end{aligned} \quad (43)$$

where $b_{1/2}^{(p)}$ depends on α_{12} and $b_{1/2}^{(q)}$ depends on α_{23} . α_{13} is a shorthand notation for $\alpha_{12}\alpha_{23}$. The first, second and third lines of Eq. (43) correspond to brackets (39), (40) and (41), respectively. The second term of the first line comes from deriving the denominators with respect to Λ_2 in Eq. (39) and would have been absent if I had factored the denominators out of the Poisson brackets.

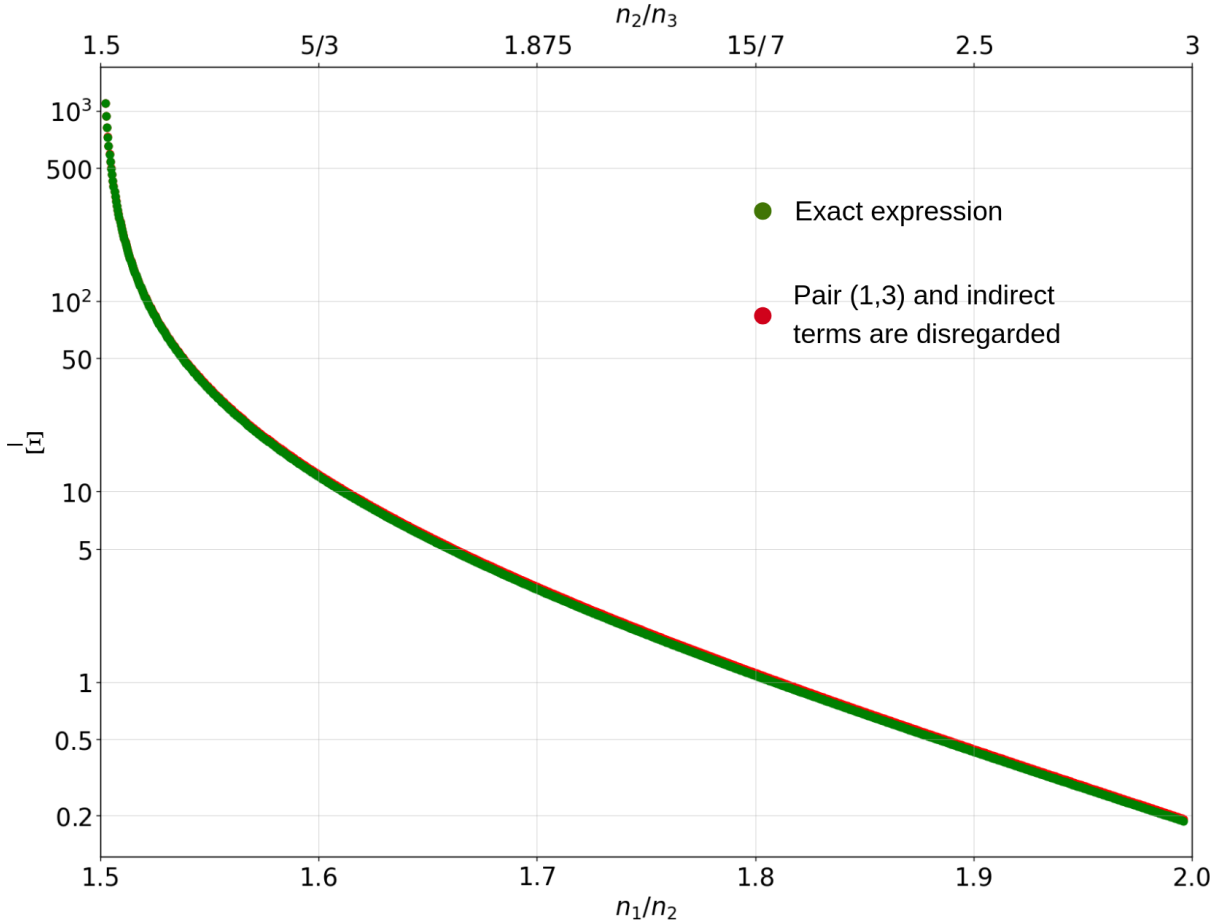


Fig. 2 — Value of $-\Xi$ on the loci of the exact resonance $2n_1 - 5n_2 + 3n_3 = 0$. The green curve is the exact value whereas the red curve is the approximation given by Eq. (43). The operator $\langle \cdot \rangle$ was defined by averaging over all 2-planet angles. Ξ diverges to $-\infty$ when both n_1/n_2 and n_2/n_3 go to $3/2$ because of denominators $(2n_1 - 3n_2)^{-1}$ and $(2n_2 - 3n_3)^{-1}$ in its expression.

The function *Check3pla* of the package verifies that the expression of Ξ given by Eq. (43) is equal to twice the output of *PerHam3pla* when called with **keplerian** = **disregard13** = **disregardInd** = True and **takeout_kn** = False. Equation (43) comes as a correction to Eq. (23) of Quillen, 2011 and Eq. (42) of Petit *et al.*, 2020, which are incorrect. In particular, Quillen disregards brackets (40) and (41), whereas Petit disregards brackets (41).

In Fig. 2, I plot the coefficient Ξ appearing in Eq. (38) in the case $(p, q) = (2, 3)$, on the loci of the exact Keplerian resonance $2n_1 - 5n_2 + 3n_3 = 0$, for $3/2 < n_1/n_2 < 2$ ($3/2 < n_2/n_3 < 3$). I compare the exact expression obtained without approximations with Eq. (43).

References

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