A FULLY ANALYTIC LUNAR THEORY

Jérémy Couturier

June 25, 2019

Supervisor : Jacques Laskar
Referring professors : Olivier Bonnefoy & Matthias Gonzalez
Institut de Mécanique Céleste et de Calcul des Éphémérides
77 avenue Denfert Rochereau, 75014, Paris

Contents

Abstract

In the 19th century, Charles-Eugène Delaunay studied the Earth-Moon-Sun three body problem in order to give analytical expressions of the elliptical elements of the Moon up to high orders (Delaunay, 1860). He limited his study to the restricted problem, where the Moon is a massless particle whose only perturbation is the sun. This work, which represents a huge amount of calculation, took him 20 years.

Last year, Li Zejian focused his internship supervised by Jacques Laskar to this same problem, doing calculations not by hand like Delaunay did but with the help of the dedicated tool *TRIP* (Gastineau and Laskar, 2011), a computer algebra system. He used the Lie formalism and *TRIP* in order to achieve the same goal as Delaunay in a few months time (LI, 2018).

In this study, I pushed further the work of Li Zejian, going back to the unrestricted problem and adding tidal interactions and dissipations in my model. The Lie formalism being extensively used throughout this study, the first section aims to set the mathematical tools needed to cope with the rest of the work. In section two, the Lie formalism is applied to the case of the simple pendulum. The section three deals with the Three-Body Problem and finally, section four adds tidal interaction in the model.

1 Lie formalism

1.1 Lie derivative

Let $\mathcal{H}(J,\theta)$ be a n-degree-of-freedom hamiltonian and f a time-independant function. The Lie derivative is then defined by :

$$L_{\mathcal{H}} f = \{\mathcal{H}, f\} := \sum_{i=1}^{n} \frac{\partial \mathcal{H}}{\partial J_i}\frac{\partial f}{\partial \theta_i} - \frac{\partial \mathcal{H}}{\partial \theta_i}\frac{\partial f}{\partial J_i} \tag{1}$$

The Hamilton-Jacobi equations read as :

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial J} \qquad \dot{J} = -\frac{\partial \mathcal{H}}{\partial \theta} \tag{2}$$

Thus $L_{\mathcal{H}} f = \displaystyle\sum_{i=1}^{n} \dot{\theta}_i \frac{\partial f}{\partial \theta_i} + \dot{J}_i \frac{\partial f}{\partial J_i} = \frac{df}{dt}\Big|_{\mathcal{H}}$ is the total time derivative of f along the trajectories of $\mathcal{H}$.

1.2 Subsequent change of variables

Let us give a n-degree-of-freedom Hamiltonian $\mathcal{H}(J,\theta) = \mathcal{H}_0(J) + \mathcal{H}_1(J,\theta)$ where $J = (J_1, ..., J_n)$ is the action variable and $\theta = (\theta_1, ..., \theta_n)$ the angular one and $\mathcal{H}_1 \ll \mathcal{H}_0$.

If we neglect $\mathcal{H}_1$ with respect to $\mathcal{H}_0$, we end up with $J(t) = J_0$ and $\theta(t) = \theta_0 + \omega_0 t$ where $\omega_0 = \dfrac{\partial \mathcal{H}_0}{\partial J}$ is the frequency on the torus $\mathbb{T}^n$. Nevertheless, let's assume that $\mathcal{H}_1$ is not small enough to do so. We would like to make a canonical change of variable $x = (\theta, J) \to x' = (\theta', J')$ such that $\mathcal{H}'(J', \theta') = \sum_{k=0}^{N} \mathcal{H}'_k(J') + \mathcal{H}^*(J', \theta')$ and the non-integrable part of $\mathcal{H}'$ is even more negligible, that is:

$$\frac{\mathcal{H}^*}{\sum_{k=0}^{N} \mathcal{H}'_k} \ll \frac{\mathcal{H}_1}{\mathcal{H}_0}$$

We write $x = \Phi_1^W(x')$ where $W = \sum_{k=1}^{N} W_k$ is a generator that we will constrain and Φ_1^W is the flow at time one of this generator. As the flow of a hamiltonian is canonical, so is this change of variables. The flow at time t of a generator W is given by the Taylor formula. Let $f \in C^\infty\left(\mathbb{R}^{2n} \to \mathbb{R}\right)$, then

$$f \circ \Phi_t^W = \sum_{n=0}^{+\infty} \frac{t^n}{n!} \frac{d^n f}{dt^n}\Big|_W = \sum_{n=0}^{+\infty} \frac{t^n}{n!} L_W^n f = e^{tL_W} f \tag{3}$$

Thus we have :

$$\mathcal{H}'(x') = \mathcal{H}(x) = \mathcal{H}\left(\Phi_1^W(x')\right) = e^{L_W} \mathcal{H}(x') \tag{4}$$

Up to the order 2, this gives $\mathcal{H}' = \mathcal{H}'_0 + \mathcal{H}'_1 + \mathcal{H}'_2$ with :

$$\begin{aligned}
\mathcal{H}'_0 &= \mathcal{H}_0 \\
\mathcal{H}'_1 &= \mathcal{H}_1 + \{W_1, \mathcal{H}_0\} \\
\mathcal{H}'_2 &= \mathcal{H}_2 + \{W_1, \mathcal{H}_1\} + \{W_2, \mathcal{H}_0\} + \frac{1}{2}\{W_1, \{W_1, \mathcal{H}_0\}\}
\end{aligned}$$

Up to the N^{th} order, we can write, for any k the following cohomological equation

$$\{\mathcal{H}_0, W_k\} = \Psi_k - \mathcal{H}'_k \tag{5}$$

where the Ψ_k are given by Deprit's recursion formula (Deprit, 1969)

$$\begin{aligned}
\Psi_1 &= \mathcal{H}_1 \\
\Psi_k &= \mathcal{H}_k + \sum_{j=1}^{k-1} L_{W_j} \mathcal{H}_{k-j} + \sum_{j=2}^{k} \frac{1}{j!} \varphi_k^j \\
\varphi_l^0 &= \mathcal{H}_l \\
\varphi_l^k &= \sum_{j=1}^{l-k+1} L_{W_j} \varphi_{l-j}^{k-1} \quad (k \leq l)
\end{aligned} \tag{6}$$

As $\mathcal{H}_0$ does not depend on the angular variables, the cohomological equation can be written under the much simpler way

$$\begin{aligned}\sum_{\mu=1}^{n} \nu_\mu \frac{\partial W_j}{\partial \theta'_\mu} &= \Psi_j - \mathcal{H}'_j \\ \nu_\mu &= \frac{\partial \mathcal{H}_0}{\partial J'_\mu}\end{aligned} \tag{7}$$

Thus we end up, for each $j \in [|1, N|]$, to a partial differential equation in the generator W. However, $\mathcal{H}'_j$ is also an unknown of the problem in the cohomological equation and we need another equation to solve the problem.

1.2.1 Averaging method

According to equation 7 choosing the form of the transformed hamiltonian $\mathcal{H}'$ is equivalent to choosing the form of the generator W and thus the change of variable. It is a priori arbitrary as we could make whatever change of variable we want (as long as it is canonical, but every flow is). As we need to repel the non-integrable part of the new hamiltonian to higher orders, we make each of the $\mathcal{H}'_j$ depend only on the action variables by setting the following averaging rule :

$$\mathcal{H}'_j = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \ldots \int_0^{2\pi} \Psi_j d\theta'_1 \ldots d\theta'_n = \langle \Psi_j \rangle \tag{8}$$

1.2.2 Resolution of the cohomological equation

In the case where $n = 1$, the cohomological equation is solved by direct integration

$$W(\theta', J') = \int \frac{\Psi_j - \langle \Psi_j \rangle}{\nu} d\theta'$$

In the case where $n \geq 1$, it is solved using Fourier series. Let's write

$$\begin{aligned}\Psi_j - \langle \Psi_j \rangle &= \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \psi_k e^{ik.\theta'} \\ W_j &= \sum_{k \in \mathbb{Z}^n \setminus \{0\}} w_k e^{ik.\theta'}\end{aligned}$$

where the ψ_k and w_k are almost all zero, as they are polynomials in our case, due to the truncations. The cohomological equation 7 yields

$$w_k = \frac{\psi_k}{i\, k.\nu} \tag{9}$$

Note that the non-resonant requirement must be met, that is $\psi_k \neq 0 \Rightarrow k.\nu \neq 0$
If $\psi_k = 0$ we can take $w_k = 0$ regardless of the value of $k.\nu$

1.3 Integration of the transformed hamiltonian

Once the change of variables is performed according to the former procedure, the transformed hamiltonian reads $\mathcal{H}'(J', \theta') = \sum_{k=0}^{N} \mathcal{H}'_k(J') + \mathcal{H}^*(J', \theta')$ where $\mathcal{H}^*$ is of order $N+1$
By neglecting $\mathcal{H}^*$, and because the change of variable we made is canonical, we can write

$$
\begin{aligned}
J'(t) &= J'_0 \\
\theta'(t) &= \theta'_0 + \nu(J')t \\
\nu(J') &= \frac{\partial(\mathcal{H}' - \mathcal{H}^*)}{\partial J'}
\end{aligned}
\tag{10}
$$

and the equations of motion of the set of new variables are solved.

1.4 Expression of the old variables

In the previous subsection, we obtained the expression of the new variables (the primed ones) as a function of time. However, these variables do not carry much physical meaning and we would rather have an expression of the old variables as a function of time.
The change of variables we defined by the Lie transformation is

$$x = \Phi_1^W(x') = e^{L_W} x' \tag{11}$$

and its inverse is

$$x' = \Phi_{-1}^W(x) = e^{-L_W} x \tag{12}$$

The procedure is thus as follow :

1. Express x as a function of x' using equation 11
2. Replace x' by its time-dependant expression given by equation 10
3. Express x' as a function of x using equation 12
4. Deduce from step 3 the values of θ'_0 and J'_0
5. Replace inside the expression of x obtained at step 2 θ'_0 and J'_0 by their actual values.

1.5 Pros and cons

The Lie transformation method generally appears to converge quickly as N increases, in particular when the condition $\mathcal{H}_1 \ll \mathcal{H}_0$ is well satisfied. Nevertheless, some cases are more complicated than others and the case of the pendulum that is studied in section 2 is for example far less problem-prone than the three-body problem.

Cases where the vector $\nu = \dfrac{\partial \mathcal{H}_0}{\partial J}$ doesn't meet the non-resonance condition may exist, in particular as N grows up (resonances can exist at a certain order even though they don't at lower orders). In that case, the resonant term in the hamiltonian shouldn't be averaged but kept inside the transformed hamiltonian, making it no longer trivially integrable, and equation 10 doesn't stand anymore.

The Lie transformation could *a priori* be pushed to any value of N, but in fact, the subsequent formal serie doesn't necessarily converges. Let us consider a non integrable primary hamiltonian $\mathcal{H}$. We could apply the Lie transformation up to infinity but the formal serie that we would establish won't converge because if it did, $\mathcal{H}'$ would be integrable and no change of variable can make an integrable hamiltonian from a non-integrable one. The first terms of the serie would then decrease rapidly before starting not to decrease fast enough for the serie to converge. The easy case of the pendulum doesn't suffer this problem as $\mathcal{H}_{\text{pendulum}}$ is integrable

2 Application to the simple pendulum

2.1 Angle and angular speed as a funtion of time

The hamiltonian of the simple pendulum can be written $\mathcal{H}(L,l) = \dfrac{L^2}{2} - \omega^2 \cos l$ where $\omega^2 = \dfrac{g}{a}$ and l is the angle between the bottom and the current position.

Let us consider a pendulum in full-revolution state, that is, the initial kick L_0 given to the pendulum allows it to complete full revolutions around its axis. This condition means $\mathcal{H} = \dfrac{L_0^2}{2} - \omega^2 > \omega^2 \Rightarrow \mathcal{H}_0 > 2\mathcal{H}_1$ with $\mathcal{H}_0 = \dfrac{L^2}{2}$ and $\mathcal{H}_1 = -\omega^2 \cos l$.

Provided that the pendulum is in full-revolution state, we can apply Lie formalism to this hamiltonian. Note that the greater the initial kick the fewer iteration are needed to get good results.

2.1.1 Case $N = 1$

The case where $N = 1$ requires very few calculation and can easily be done by hand. The cohomological equation is

$$\{\mathcal{H}_0, W_1\} = \Psi_1 - \mathcal{H}'_1 \qquad \Psi_1 = \mathcal{H}_1 \qquad \mathcal{H}'_1 = \langle \Psi_1 \rangle_l$$

which yields

$$\begin{aligned} &W_1(L',l') = -\frac{\omega^2}{L'} \sin l' \qquad \mathcal{H}'(L') = \frac{L'^2}{2} \\ &L'(t) = L'_0 \qquad l'(t) = l'_0 + L'_0 t \end{aligned}$$

Now let's go back to the old variables. Applying the generator to l, l', L and L' and truncating the subsequent series to order 1 in $\dfrac{\omega^2}{L^2}$, we get

$$l(t) = l'_0 + L'_0 t + \frac{\omega^2}{L_0'^2} \sin(l'_0 + L'_0 t)$$

$$L(t) = L'_0 + \frac{\omega^2}{L'_0} \cos(l'_0 + L'_0 t)$$

With

$$l_0' = l_0 - \frac{\omega^2}{L_0^2}\sin(l_0)$$
$$L_0' = L_0 - \frac{\omega^2}{L_0}\cos(l_0)$$

2.1.2 Higher orders

When $\mathcal{H}_0 = \dfrac{L^2}{2}$ is not so great with respect to $\mathcal{H}_1 = -\omega^2\cos(l)$, we may want to go to higher values of N for a better result. The subsequent amount of calculation increases rapidly with N, even with a mere pendulum and we use *TRIP* to perform them. Results of the computation by *TRIP* for the special case $N = 5$ are given in appendix A. Once again, the infinite series generated by W are truncated at order N in $\frac{\omega^2}{L^2}$.

2.1.3 Comparison with experiment

We would like to compare the performances of the Lie method with the experiment, assumed to be a numerical integration. On the following figure, plots for $N = 1, 2, 5, 10$ and 15 are represented along with a Runge Kutta 4 numerical integration of the simple pendulum. Chosen numerical values are $\omega^2 = 0.1\ s^{-2}$, $L_0 = 0.7\ s^{-1}$ and $l_0 = 0$

This value of ω^2 implies $L_{0,crit} = \dfrac{2}{\sqrt{10}} \approx 0.63$ for the critical initial kick that enables the full-revolution state. L_0 being just above $L_{0,crit}$, we expect little values of N will lead to bad results.

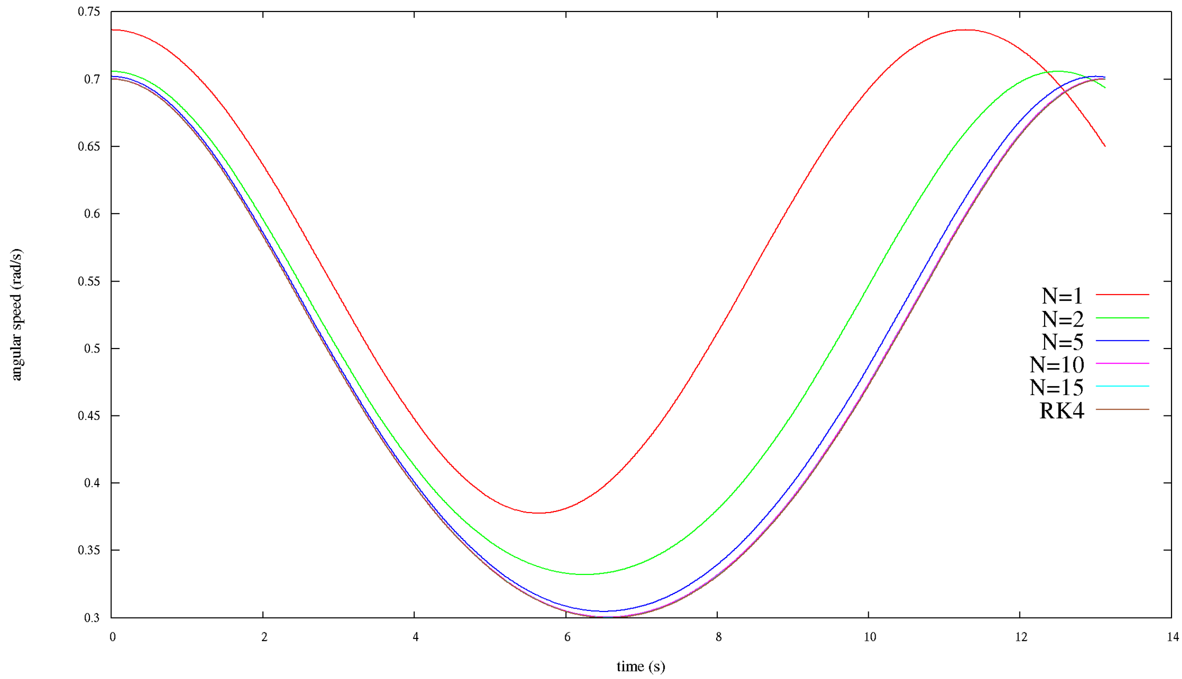


Figure 1: Angular velocity as a function of time at different orders of the Lie transformation

As expected, $N = 1$ and $N = 2$ are quite far from the numerical integration but then the method converges rapidly for higher values of N.

2.2 Mean angular velocity of a pendulum by Lie transformations

2.2.1 First establishment of the result

$\mathcal{H}$ being a constant of the motion ($\frac{d\mathcal{H}}{dt} = \{\mathcal{H}, \mathcal{H}\} = 0$), we can use its expression to obtain an expression of $L = \dot{l}$ depending only on l and then a relation between dt and dl. We get

$$dt = \frac{dl}{\sqrt{2\left(\mathcal{H} + \omega^2 \cos l\right)}}$$

From which can be deduced

$$T = \int_0^T dt = \int_0^{2\pi} \frac{dl}{\sqrt{2\left(\mathcal{H} + \omega^2 \cos l\right)}}$$

$$\frac{\sqrt{2\mathcal{H}}}{2\pi} T = \sum_{n=0}^{+\infty} \left\{ (-\epsilon)^n \left(\prod_{k=1}^{2n} \frac{2k-1}{2k} \right) \frac{1}{4^n} \binom{2n}{n} \right\} = 1 + \frac{3}{16}\epsilon^2 + \frac{105}{1024}\epsilon^4 + \frac{1155}{16\,384}\epsilon^6 + \ldots$$

Where $\epsilon = \dfrac{\omega^2}{\mathcal{H}}$
The mean angular velocity is obtained by inverting the serie of T. We end up with

$$\langle L \rangle = L_0\sqrt{1-2\kappa}\left(1 - \frac{3}{16}\epsilon^2 - \frac{69}{1024}\epsilon^4 - \frac{633}{16\,384}\epsilon^6 + ...\right)$$

Where $\kappa = \dfrac{\omega^2}{L_0^2} = \dfrac{\epsilon}{2(1+\epsilon)}$ when l_0 is set to 0 and finally

$$\langle L \rangle = L_0\left(1 - \kappa - \frac{5}{4}\kappa^2 - \frac{11}{4}\kappa^3 - ... - \frac{571\,915\,951}{65\,536}\kappa^{10} - ...\right)$$

2.2.2 Second establishment by Lie transformations

The Lie formalism also allows to reach this results. Once the expression of $L(l_0', L_0', t)$ up to the desired order is known, only non-periodic terms are kept to have the mean value of L. l_0' and L_0' are then substituted by their expression in function of l_0 and L_0 and l_0 is set to zero.

Here, all series were truncated at order 10 in the variable $\dfrac{\omega^2}{L_0^2}$, and the Lie transformation was performed up to the order N such that the subsequent generator W_N is zero (such N exists thanks to the truncation, otherwise it wouldn't) which turned out to be 11. Here is the output of *TRIP* (Note that $a = \omega^2$)

$$\langle L \rangle =$$

```
1*L0
- 1*L0**-1*a
- 5/4*L0**-3*a**2
- 11/4*L0**-5*a**3
- 469/64*L0**-7*a**4
- 1379/64*L0**-9*a**5
- 17223/256*L0**-11*a**6
- 56001/256*L0**-13*a**7
- 11998869/16384*L0**-15*a**8
- 41064827/16384*L0**-17*a**9
- 571915951/65536*L0**-19*a**10
```

We end up to the same expression as given by the straightforward calculation.

3 The three-body problem

3.1 Conventions an notations

Throughout this section, the orbital elements commonly used in celestial mechanics will appear, as well as some personal notations, some of them being inspired from the work of Li Zejian, namely :

e the eccentricity
a the semi-major axis
i the inclination of the orbital plane
Ω the longitude of the ascending node
ω the argument of the periapsis
M the mean anomaly
$\gamma = \sin(i/2)$
$\varpi = \omega + \Omega$ the longitude of the periapsis
$\lambda = \varpi + M$ the mean longitude
$\mu = Gm$ the gravitational parameter
v the true anomaly
$w = v + \varpi$ the true longitude

As a convention, unsubscripted letters will denote coordinates of the Moon while letter subscripted with an s will serve for the Sun. Vector will be written in bold characters.

Along with the aforementioned varibles, Delaunay's and Poincaré's **canonical** variables will be extensively used. Delaunay variables are

$$\begin{array}{c|l} \text{Action} & \text{Angle} \\ L = \beta\sqrt{\mu a} & l = M \\ G = L\sqrt{1-e^2} & g = \omega \\ H = G\cos i & h = \Omega \end{array}$$

While Poincaré's variables are

$$\begin{array}{c|c} \text{Action} & \text{Angle} \\ \Lambda = \beta\sqrt{\mu a} & \lambda = M + \varpi \\ \Gamma = \Lambda\left(1 - \sqrt{1-e^2}\right) & \gamma = -\varpi \\ Z = \Lambda\sqrt{1-e^2}(1-\cos i) & z = -\Omega \end{array}$$

The canonicity of Delaunay's variables can be shown using Andoyer formalism (Andoyer, 1923). Then, the canonicity of Poincaré's variables is almost immediate as they are obtained by a linear transformation of the actions in Delaunay's variables. We have

$$\begin{pmatrix} \Lambda \\ \Gamma \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} L \\ G \\ H \end{pmatrix} = A \begin{pmatrix} L \\ G \\ H \end{pmatrix}$$

To be canonical, the change of variables must comply with

$$\begin{pmatrix} \lambda \\ \gamma \\ z \end{pmatrix} = {}^tA^{-1} \begin{pmatrix} l \\ g \\ h \end{pmatrix}$$

yielding the above result.

In all the following, Poincaré's variables will be preferred to Delaunay's as they are more suitable at small eccentricity and inclination. Nevertheless, they will be denoted with the same name as Delaunay's variables for convenience, yielding no ambiguity as we will not use Delaunay's variables in this study. Therefore we will have, for the rest of this work

$$
\begin{array}{r|l}
\text{Action} & \text{Angle} \\
L = \beta\sqrt{\mu a} & l = M + \varpi \\
G = L\left(1 - \sqrt{1-e^2}\right) & g = -\varpi \\
H = L\sqrt{1-e^2}(1 - \cos i) & h = -\Omega
\end{array}
\tag{13}
$$

Because they are unused in the restricted problem, values of β and β_s will be further established, in section 3.3

3.2 The restricted problem

3.2.1 Establishment of $\mathcal{H}$

Treating the Moon as a massless particle allows to greatly simplify the problem as the Earth-Sun system can be treated independently and is no more than a two-body problem.

In the geocentric reference frame, the equation of motion of the Moon writes

$$
\begin{aligned}
&\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mu}{r^3}\mathbf{r} - \mu_s\left(\frac{\mathbf{r_s}}{r_s^3} - \frac{\mathbf{r_s} - \mathbf{r}}{|\mathbf{r_s} - \mathbf{r}|^3}\right) \\
&\mu = G\left(m_E + m_M\right) \qquad \mu_s = Gm_s
\end{aligned}
\tag{14}
$$

The hamiltonian of the problem can be deduced from equation 2

$$
\begin{aligned}
&\mathcal{H}(\tilde{\mathbf{r}}, \mathbf{r}) = \frac{\tilde{\mathbf{r}}^2}{2} - \frac{\mu}{r} - \mu_s\left(\frac{1}{|\mathbf{r_s} - \mathbf{r}|} - \frac{\mathbf{r_s}.\mathbf{r}}{r_s^3}\right) = \mathcal{H}_{\text{kepler}} + \mathcal{H}_1 \\
&\mathcal{H}_{\text{kepler}} = \frac{\tilde{\mathbf{r}}^2}{2} - \frac{\mu}{r} \qquad \mathcal{H}_1 = -\mu_s\left(\frac{1}{|\mathbf{r_s} - \mathbf{r}|} - \frac{\mathbf{r_s}.\mathbf{r}}{r_s^3}\right)
\end{aligned}
\tag{15}
$$

As $\mathbf{r_s}$ is time-dependent, so is $\mathcal{H}_1$ and we need to go to the extended space phase. This can be achieved by noticing that in the restricted problem, the mean longitude of the sun l_s is a linear function of time. Noting K the conjugate momentum of l_s, we note $\mathcal{H}_0 = \mathcal{H}_{\text{kepler}} + \nu_k K$, ν_k being the mean motion of the sun, a parameter of the restricted problem.

At this stage, our hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ is time independent and comply with $\mathcal{H}_1 \ll \mathcal{H}_0$, but $\mathcal{H}_0$ doesn't depend only on the conjugate momentum and we can't apply the Lie formalism presented in section 1. Using the canonical variables of Poincaré will nevertheless solve this issue. First note that in the restricted problem, L, the action variable associated to l, is $\sqrt{\mu a}$

In Poincaré's variable, we have $\mathcal{H}_0 = -\dfrac{\mu^2}{2L^2} + \nu_k K$ which depends only on the action variable. All we have left to do is to express $\mathcal{H}_1$ using Poincaré's variables.

3.2.2 From cartesian variables to Poincaré's

Let us define $\nu = \dfrac{\sqrt{\mu_s}}{a_s\sqrt{a_s}}$ and $n = \dfrac{\sqrt{\mu}}{a\sqrt{a}}$. We know, thanks to the two-body problem, that n is the instantaneous mean motion of the Moon. However, ν is only **almost** the mean motion of the sun around the Earth as $\mu_s = Gm_s \neq G\,(m_E + m_s)$.

Defining S as the angle between $\mathbf{r}$ and $\mathbf{r_s}$, $\mathcal{H}_1$ can now be written as

$$
\begin{aligned}
\mathcal{H}_1 &= -\nu^2 a^2 \left(\frac{r}{a}\right)^2 \left(\frac{a_s}{r_s}\right)^3 \left(\frac{r}{r_s}\right)^{-2} \left[\frac{1}{\sqrt{1+\frac{r^2}{r_s^2} - 2\frac{r}{r_s}\cos S}} - \frac{r}{r_s}\cos S - 1\right] \\
\mathcal{H}_1 &= -\nu^2 a^2 \left(\frac{r}{a}\right)^2 \left(\frac{a_s}{r_s}\right)^3 \left(\frac{r}{r_s}\right)^{-2} \sum_{n=2}^{+\infty} \left(\frac{r}{r_s}\right)^n P_n\left(\cos S\right)
\end{aligned}
\tag{16}
$$

Where the P_n are the Legendre polynomials.

$\dfrac{r}{a}$ and $\dfrac{a_s}{r_s}$ can be developed in power series of e and e^{iM} thanks to the following expressions (Laskar, 2007)

$$
\begin{aligned}
\frac{a}{r} &= \sum_{k=-\infty}^{+\infty} J_k(ke) e^{ikM} \\
\frac{r}{a} &= 1 - e\cos\left(M + 2\sum_{k=1}^{+\infty} \frac{J_k(ke)}{k}\sin(kM)\right) \\
J_k(x) &= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{k+2n}}{2^{k+2n}(k+n)!\,n!}
\end{aligned}
\tag{17}
$$

Where the J_k are the first kind Bessel's functions.
TRIP's incorporated functions **RsA** and **AsR**, which give such development were used for this part.
The $\cos S$ can be computed thanks to (Laskar and Robutel, 1995) and the *TRIP* function **Expiv** which gives the development of $e^{iv} = e^{i(w+g)}$. We have

$$
\begin{aligned}
v - M &= 2\sum_{k=1}^{+\infty} \frac{1}{k}\left(\sum_{l=-\infty}^{+\infty} \rho^{|k-l|} J_l(ke)\right)\sin kM \\
\cos S &= \mathfrak{R}_e\left\{ e^{i(w+w_s)}\left(\bar{\xi}\cos\frac{i_s}{2} - \bar{\xi}_s\cos\frac{i}{2}\right)^2 + e^{i(w-w_s)}\left(\cos\frac{i}{2}\cos\frac{i_s}{2} + \bar{\xi}\xi_s\right)^2 \right\} \\
\xi &= \gamma e^{i\Omega} = \sin\frac{i}{2}\,e^{i\Omega} \qquad \rho = \frac{1-\sqrt{1-e^2}}{e} = \frac{e}{2} + \frac{e^3}{8} + \mathcal{O}(e^5)
\end{aligned}
\tag{18}
$$

All is left to do is to write $\dfrac{r}{r_s} = \alpha\dfrac{r}{a}\dfrac{a_s}{r_s}$ where $\alpha = \dfrac{a}{a_s}$ then, equations 17 and 18 enable to write $\mathcal{H}_1$ as a function of l, g, h, e, α and γ

As the plane of the orbit of the Sun around the Earth is constant, we can set $i_s = 0$ and therefore $\gamma_s = 0$ without loss of generality, yielding a lot of simplification.

As the action variables don't appear into $\mathcal{H}_1$, we must define the derivative with respect to them. Let us denote $\eta = \frac{\nu}{n}$ We have

$$\begin{aligned}
\frac{\partial}{\partial L} &= \frac{\partial \gamma}{\partial L}\frac{\partial}{\partial \gamma} + \frac{\partial \eta}{\partial L}\frac{\partial}{\partial \eta} + \frac{\partial \alpha}{\partial L}\frac{\partial}{\partial \alpha} + \frac{\partial e}{\partial L}\frac{\partial}{\partial e} \\
\frac{\partial}{\partial G} &= \frac{\partial e}{\partial G}\frac{\partial}{\partial e} + \frac{\partial \gamma}{\partial G}\frac{\partial}{\partial \gamma} \\
\frac{\partial}{\partial H} &= \frac{\partial \gamma}{\partial H}\frac{\partial}{\partial \gamma}
\end{aligned} \tag{19}$$

Where, from relations 13

$$\begin{aligned}
\frac{\partial \gamma}{\partial L} &= -\frac{\eta\gamma}{2a_s^2\alpha^2\sqrt{1-e^2}\nu} \\
\frac{\partial \alpha}{\partial L} &= \frac{2\eta}{a_s^2\alpha\nu} \\
\frac{\partial \eta}{\partial L} &= \frac{3\eta^2}{a_s^2\alpha^2\nu} \\
\frac{\partial e}{\partial L} &= \frac{\eta\left(1-e^2-\sqrt{1-e^2}\right)}{a_s^2\alpha^2 e\nu} \\
\frac{\partial e}{\partial G} &= \frac{\eta\sqrt{1-e^2}}{a_s^2\alpha^2 e\nu} \\
\frac{\partial \gamma}{\partial G} &= \frac{\eta\gamma}{2a_s^2\alpha^2\sqrt{1-e^2}\nu} \\
\frac{\partial \gamma}{\partial H} &= \frac{\eta}{4a_s^2\alpha^2\sqrt{1-e^2}\gamma\nu}
\end{aligned} \tag{20}$$

3.2.3 Lie transformation

The hamiltonian being written as a function of the elliptical elements and the derivative with respect to the action variable being defined, we can now perform Lie transformations.

All series are truncated at some order N on the total degree of the little quantities of the problem, namely e, e_s, γ, α and η. That is, all terms of the form $e^x e_s^y \gamma^z \alpha^t \eta^u$ such that $x + y + z + t + u > N$ are got rid of. Please note that even though it wasn't mentioned, the same truncation rule was also set to go from cartesian variables to elliptical elements.

Lie transformations are then performed up to the order N' such that the subsequent generator W'_N is zero. The setting of the truncation rule makes sure that such N' actually exists. Generally, N' is not much greater than N.

One can notice that $\mathcal{H}_0 = -\dfrac{\alpha^2 a_s^2 \nu^2}{2\eta^2} + \nu_k K$ only depends on L and L_s. This is quite problematic as the frequencies $\nu_G = \dfrac{\partial \mathcal{H}_0}{\partial G}$ and $\nu_H = \dfrac{\partial \mathcal{H}_0}{\partial H}$ that appear in the cohomological equation 7 will always be zero.

This issue is solved by making the Lie transformation in two steps.

Step 1

In the first step, only l and l_s are averaged in equation 8. This leads to a generator W' and a transformed hamiltonian $\mathcal{H}' = \mathcal{H}'_0 + \mathcal{H}'_1$. After this step is complete, $\mathcal{H}'$ no longer depends on l and ls, thus L and L_s are now constants of the motion and so is $\mathcal{H}'_0$, which can be removed from $\mathcal{H}'$ without changing the form of the Hamilton-Jacobi equations.

Step 2

For the step 2, $\mathcal{H}'_0$ is defined as the lowest-orders term with dependence on G' and H' in $\mathcal{H}'$, which turns out to be $\mathcal{H}'_0 = a_s^2 \nu^2 \alpha'^2 \left(\frac{3}{2}\gamma'^2 - \frac{3}{8}e'^2\right)$. $\mathcal{H}'_1$ is then set to $\mathcal{H}' - \mathcal{H}'_0$. The Lie transformation is now performed by averaging g and h in equation 8. This leads to a generator W'' and a transformed hamiltonian $\mathcal{H}''$ which doesn't depend on the angular variables (thanks to the truncation) and is thus trivially integrable.

For the going back into the old variables, one has to be cautious as W' and W'' don't commute and nor do $\Phi_1^{W'}$ and $\Phi_1^{W''}$.

The change of variable from old to new ones and from new to old ones can be deduced from equation 3. We have, denoting $x = (a, e, \gamma, K, l, g, h, l_s)$

$$\begin{aligned} x &= \Phi_1^{W'} \circ \Phi_1^{W''}(x'') = e^{L_{W''}}\left(\Phi_1^{W'}\right)(x'') = e^{L_{W''}} e^{L_{W'}} x'' \\ x'' &= \Phi_{-1}^{W''} \circ \Phi_{-1}^{W'}(x) = e^{-L_{W'}}\left(\Phi_{-1}^{W''}\right)(x) = e^{-L_{W'}} e^{-L_{W''}} x \end{aligned} \tag{21}$$

3.3 The unrestricted problem

3.3.1 Establishment of $\mathcal{H}$ in Jacobi's variables

The case of the unrestricted problem is trickier as going into the geocentric reference frame and deriving the hamiltonian from the equation of motion doesn't yield a canonical hamiltonian. The Moon having recovered its mass, considering the center of the Earth as a privileged point is no longer relevant and the barycenter of the Earth-Moon system should be regarded instead.

In an inertial reference frame, the hamiltonian of the three-body problem is

$$\mathcal{H}(\tilde{\mathbf{u}}, \mathbf{u}) = \frac{\tilde{u}_0^2}{2m_0} + \frac{\tilde{u}_1^2}{2m_1} + \frac{\tilde{u}_2^2}{2m_2} - G\left[\frac{m_0 m_1}{|\mathbf{u_1} - \mathbf{u_0}|} + \frac{m_0 m_2}{|\mathbf{u_2} - \mathbf{u_0}|} + \frac{m_1 m_2}{|\mathbf{u_2} - \mathbf{u_1}|}\right] \tag{22}$$

In order to identify the positions of body n°1 and body n°2 with respect to body n°0 and the barycenter of bodies n°0 and 1, respectively, let us go into Jacobi's variables by performing the following transformation

$$\begin{pmatrix} \mathbf{r_0} \\ \mathbf{r_1} \\ \mathbf{r_2} \end{pmatrix} = \begin{pmatrix} \frac{m_0}{M} & \frac{m_1}{M} & \frac{m_2}{M} \\ -1 & 1 & 0 \\ -(1-\delta) & -\delta & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u_0} \\ \mathbf{u_1} \\ \mathbf{u_2} \end{pmatrix} = A \begin{pmatrix} \mathbf{u_0} \\ \mathbf{u_1} \\ \mathbf{u_2} \end{pmatrix} \qquad \delta = \frac{m_1}{m_0 + m_1} \qquad M = m_0 + m_1 + m_2 \tag{23}$$

$\mathbf{r_0}$ now points toward the barycenter of the system, $\mathbf{r_1}$ gives the position of body 1 with respect to body 0 and $\mathbf{r_2}$ gives the positions of body 2 with respect to the barycenter of body 0 and 1.

For this transformation to be canonical, the associated momentums must verify

$$\tilde{\mathbf{u}} = {}^{t}A\,\tilde{\mathbf{r}} \tag{24}$$

In this new coordinates, the hamiltonian of the system reads

$$\begin{aligned} \mathcal{H}(\tilde{\mathbf{r}}, \mathbf{r}) &= \frac{\tilde{r}_0^2}{2M} + \frac{\tilde{r}_1^2}{2\beta_1} + \frac{\tilde{r}_2^2}{2\beta_2} - G\left[\frac{m_0 m_1}{r_1} + \frac{m_0 m_2}{|\mathbf{r_2} + \delta\mathbf{r_1}|} + \frac{m_1 m_2}{|\mathbf{r_2} - (1-\delta)\mathbf{r_1}|}\right] \\ \beta_1 &= \frac{m_0 m_1}{m_0 + m_1} \qquad \beta_2 = \frac{m_2(m_0 + m_1)}{M} \end{aligned} \tag{25}$$

This hamiltonian doesn't depend on $\mathbf{r_0}$, thus, $\tilde{\mathbf{r}}_\mathbf{0}$ is a constant of motion and can be removed from the hamiltonian. This traduces the conservation of the total kinetic energy of the system.

From now on, body 1 will be the Moon, located with respect to the center of the Earth and body 2 will be the Sun, located with respect to the Earth-Moon barycenter.

3.3.2 From Jacobi's variables to Poincaré's

From the following relations

$$\begin{aligned} &\frac{\tilde{r}^2}{2\beta} - \frac{G m_e m}{r} = -\frac{\mu^2 \beta^3}{2L^2} \\ &\frac{\tilde{r}_s^2}{2\beta_s} - \frac{G(m_e + m) m_s}{r_s} = -\frac{\mu_s^2 \beta_s^3}{2L_s^2} \\ &\mu = G(m + m_e) \qquad \mu_s = GM \end{aligned} \tag{26}$$

where m_e is the mass of the Earth, $\mathcal{H}$ can be written as

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_0 + \mathcal{H}_1 \\ \mathcal{H}_0 &= -\frac{\mu^2 \beta^3}{2L^2} - \frac{\mu_s^2 \beta_s^3}{2L_s^2} \\ \mathcal{H}_1 &= \frac{G(m + m_e) m_s}{r_s} - \frac{G m_e m_s}{|\mathbf{r_s} + \delta\mathbf{r}|} - \frac{G m m_s}{|\mathbf{r_s} - (1-\delta)\mathbf{r}|} \end{aligned} \tag{27}$$

$\mathcal{H}_0$ corresponds to the hamiltonian of two independent Kepler's problems. We expressed it as a function of the Poincaré variables and it only depends on the action variables. As for $\mathcal{H}_1$, it should be written

$$\mathcal{H}_1 = -\frac{G\beta m_s}{r_s}\sum_{n=2}^{+\infty}\left\{(1-\delta)^{n-1} - (-1)^{n-1}\delta^{n-1}\right\}\left(\frac{r}{r_s}\right)^n P_n(\cos S) \tag{28}$$

Where the P_n are the Legendre polynomials and $\cos S$ is the angle between $\mathbf{r}$ and $\mathbf{r_s}$

Written like this, $\mathcal{H}_1$ can be expressed in terms of the orbital elements using equations 17 and 18

Let us now define the following dimensionless quantities that will be useful to set truncation rules.

$$x = \sqrt{\frac{G}{L}} \quad x_s = \sqrt{\frac{G_s}{L_s}} \quad y = \sqrt{\frac{H}{L}} \quad y_s = \sqrt{\frac{H_s}{L_s}} \quad \eta = \sqrt{\frac{L}{L_s}} \tag{29}$$

From relations

$$\begin{aligned} &e = \sqrt{2}\sqrt{1-x^2}x \quad e_s = \sqrt{2}\sqrt{1-x_s^2}x_s \\ &\gamma = \frac{y}{\sqrt{2}}\sqrt{\frac{1}{1-x^2}} = \sin\frac{i}{2} \quad \gamma_s = \frac{y_s}{\sqrt{2}}\sqrt{\frac{1}{1-x_s^2}} = \sin\frac{i_s}{2} \end{aligned} \tag{30}$$

We can express the hamiltonian $\mathcal{H}_1$ as a function of $x, y, x_s, y_s, \eta, L_s, l, g, h, l_s, g_s$ and h_s. Like in section 3.2, the derivative with respect to the action variables should be defined as they don't appear in $\mathcal{H}_1$. We have

$$\begin{aligned} &\frac{\partial}{\partial L} = \frac{\partial x}{\partial L}\frac{\partial}{\partial x} + \frac{\partial y}{\partial L}\frac{\partial}{\partial y} + \frac{\partial \eta}{\partial L}\frac{\partial}{\partial \eta} \qquad \frac{\partial}{\partial G} = \frac{\partial x}{\partial G}\frac{\partial}{\partial x} \qquad \frac{\partial}{\partial H} = \frac{\partial y}{\partial H}\frac{\partial}{\partial y} \\ &\frac{\partial}{\partial L_s} = \frac{\partial x_s}{\partial L_s}\frac{\partial}{\partial x_s} + \frac{\partial y_s}{\partial L_s}\frac{\partial}{\partial y_s} + \frac{\partial \eta}{\partial L_s}\frac{\partial}{\partial \eta} + \frac{\partial}{\partial L_s} \\ &\frac{\partial}{\partial G_s} = \frac{\partial x_s}{\partial G_s}\frac{\partial}{\partial x_s} \qquad \frac{\partial}{\partial H_s} = \frac{\partial y_s}{\partial H_s}\frac{\partial}{\partial y_s} \end{aligned} \tag{31}$$

And

$$\begin{aligned} &\frac{\partial x}{\partial L} = -\frac{x}{2\eta^2 L_s} \qquad \frac{\partial y}{\partial L} = -\frac{y}{2\eta^2 L_s} \qquad \frac{\partial \eta}{\partial L} = \frac{1}{2\eta L_s} \\ &\frac{\partial x}{\partial G} = \frac{1}{2x\eta^2 L_s} \qquad \frac{\partial y}{\partial H} = \frac{1}{2y\eta^2 L_s} \\ &\frac{\partial x_s}{\partial L_s} = -\frac{x_s}{2L_s} \qquad \frac{\partial y_s}{\partial L_s} = -\frac{y_s}{2L_s} \qquad \frac{\partial \eta}{\partial L_s} = -\frac{\eta}{2L_s} \\ &\frac{\partial x_s}{\partial G_s} = \frac{1}{2x_s L_s} \qquad \frac{\partial y_s}{\partial H_s} = \frac{1}{2y_s L_s} \end{aligned} \tag{32}$$

The hamiltonian $\mathcal{H}_1$ was obtained using *TRIP* and the previous results. The *TRIP* source code that does so is displayed in appendix B. As for the explicit expression of $\mathcal{H}_1$, it is displayed in appendix C.

Our hamiltonian being now expressed in terms of the Poincaré variables and the quantities defined in relations 29, we are ready to perform Lie transformations in the same way as we did in section 3.2.3, but before we do so, let's define precise truncation rules.

3.3.3 Truncation rule

Truncations will be performed on the small dimensionless quantities (constants as well as variables) of the problem, namely $x, y, x_s, y_s, \eta, \xi = \dfrac{\beta}{\beta_s}, \delta = \dfrac{m}{m + m_p}$ and $\kappa = \dfrac{\mu}{\mu_s}$. The chosen plane of reference is the plane perpendicular to the total angular momentum of the system. The value at time $t =$ J2000 (01/01/2000) of this quantities being given here for reference.

$$
\begin{array}{llll}
x_0 \approx \dfrac{1}{22.4} & y_0 \approx \dfrac{1}{15.5} & x_{s0} \approx \dfrac{1}{85} & y_{s0} \approx \dfrac{1}{14\,671\,300} \\
\eta_0 \approx \dfrac{1}{967} & \delta \approx \dfrac{1}{82} & \xi \approx \dfrac{1}{81} & \kappa \approx \dfrac{1}{332\,946}
\end{array}
$$

As the Moon only contributes for a very small amount in the total angular momentum, the sun almost orbits in the reference plane, hence the very small value of y_s. Despite its small value, the sun inclination is still large enough for its node longitude to be defined. From now on, we will truncate every serie that appear in our problem to a certain total order N.

We considered the quantities $x, y, x_s, y_s, \eta, \xi, \delta$ and κ to be of order $1, 1, 1, 5, 2, 1$ and 4 respectively. Once the total order N defined, Lie transformations were performed up to the order such that subsequent generators are zero.

3.3.4 Lie transformation

To make our hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ integrable, we are going to use the Lie formalism described in section 1. Similarly as done in section 3.2, the change of variable is done in two step.

In first step, only the mean motion of the Moon and the Sun (l and l_s) are averaged in equation 8. After this first step, $\mathcal{H}_0'$ is constant in the new system of coordinates as it only depends on L' and L_s' and $\mathcal{H}'$ doesn't depend on l' and l_s'

$\mathcal{H}_0'$ is thus removed from the hamiltonian and the lowest order term with dependence on G', H', G_s' and H_s' in $\mathcal{H}_1'$ is removed from $\mathcal{H}_1'$ and put in $\mathcal{H}_0'$. This term was, for $N \geq 7$:

$$
\mathcal{H}_0' = \frac{3}{4} \frac{\beta_s^3 \mu_s^2 \delta \eta^8}{\xi^4 \kappa^2 L_s^2} \left(y^2 - x^2 - x_s^2 \right)
$$

We see that this term doesn't depend on y_s and thus on H_s. In fact, due to our truncation rule and the high order of y_s, we never have dependence on y_s in $\mathcal{H}_0$. This prevents us from averaging over h_s

Lie transformations are thus repeated with average over g', h' and g_s'.

After step two is complete, we end up with a hamiltonian $\mathcal{H}''$ which only depends on the action variables and h_s'', a priori. In reality, even at the biggest order we went to, the average over g, h and g_s removed the dependence over h_s. Our new hamiltonian $\mathcal{H}''$ is thus integrable.

We now know the expression of the double-primed variables as a function of time thanks to equations 2 and we can go back to the unprimed variables using equation 21

As a compromise between quality of the model and runtime, we went up to $N = 13$. For such N, the first Lie transformation was performed up to the order 8 and the second one up to order 6 for the subsequent generators to be zero. The total runtime (Expression of $\mathcal{H}$ in Poincaré variables, two successive change of variable and computation of $\mathcal{H}''$, expression of the unprimed variables as a function of the double-primed ones and expression of the double-primed variables as a function of the unprimed ones) was about 40 hours on the dedicated cluster of the IMCCE (Institute for Celestial Mechanics and Calculations of Ephemeris). Here is shown the size (the number of terms) of the series that were manipulated and computed by *TRIP*

Serie	9	11	13	Serie	9	11	13
$\mathcal{H}$	539	2624	9612	$\mathcal{H}''$	33	126	405
W	446	4100	25214	W'	4	52	328
ν_L	32	125	401	y_s''	1	632046	4997627
ν_G	9	45	167	η''	1948	14956	78700
ν_H	9	46	172	L_s''	427	3996	24646
ν_{L_s}	32	125	404	l	17087	107434	464102
ν_{G_s}	6	31	124	g	287556	1687430	7014948
ν_{H_s}	1	1	1	h	54148	414101	1890109
l''	16989	106177	460659	l_s	477	4264	26199
g''	298525	1736739	7262313	g_s	1443	14030	84943
h''	65243	464631	2062297	h_s	1	4118963	30365317
l_s''	447	4127	25399	x	171129	1100612	5031341
g_s''	1431	13433	79615	y	25805	203609	992127
h_s''	1	8207025	35360647	x_s	554	6031	40148
x''	172616	1100322	5039100	y_s	1	292898	4407713
y''	29176	216518	1044233	η	2318	18912	102889
x_s''	562	6014	39076	L_s	414	3914	24755

At order 13, the total size of all the stored series was 19.8 Go.

The *TRIP* source code that performed the Lie transformation and went back into the unprimed variables to obtain a time dependent expression of the orbital elements is given in appendix D

In appendix E is displayed the transformed hamiltonian $\mathcal{H}''$ as well as the generators W and W'

3.3.5 Comparison with Experiment

In this subsection, we compare our analytical series up to different orders with the numerical simulation ran by Hervé Manche at the IMCCE, using ADAMS and the INPOP integrator, used to compute ephemeris. He set the mass of all bodies of the solar system but the Sun's, the Earth's and the Moon's to 0 and he ran the simulation with a time step of $dt \approx 1.3282$ hours. He supplied me with a file of the orbital elements from $t = 0$ to $t = 20$ years with a small time step and another file from $t = 0$ to $t = 100\,000$ years with a larger time step to check the validity over large time of the analytical solution. To check the validity of the numerical integration, Hervé Manche made sure that the total energy of the system remained the same over time. The error after 100 000 years doesn't exceed 2.5×10^{-18}, assuring the reliability of the simulation.

Here is shown the evolution of the excentricity of the Moon and the Sun over a period of one year.

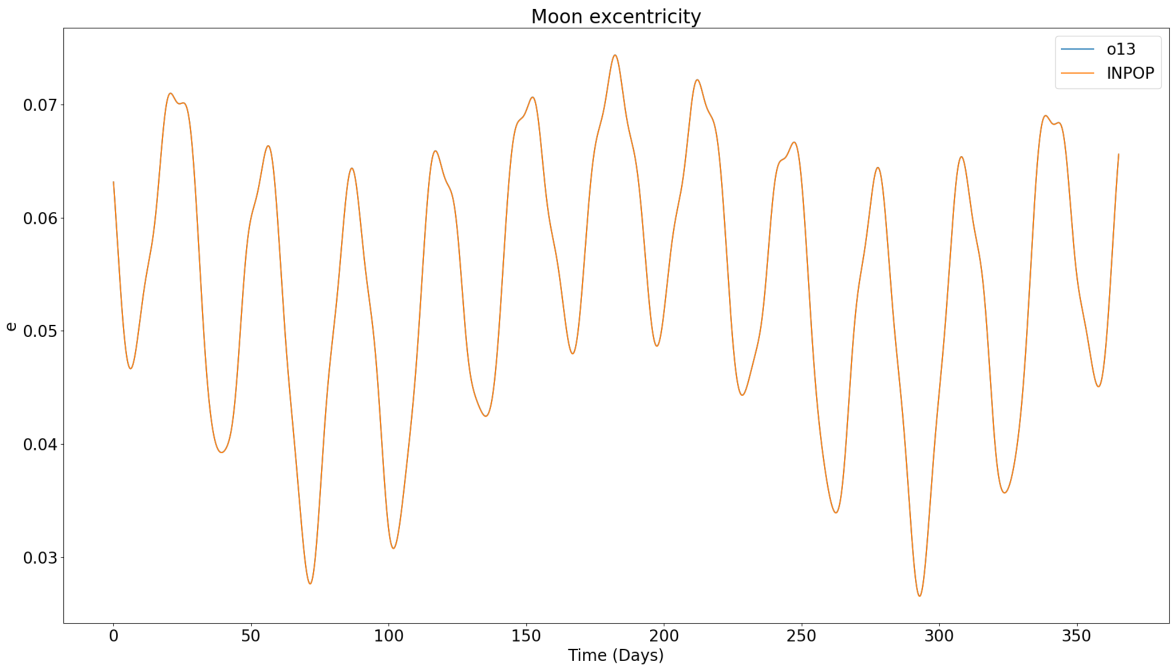


Figure 2: Excentricity of the Moon from J2000 to J2000+ 1 year

Even though they are very small, the Sun undergoes excentricity variations too.

In both cases, the INPOP simulation and the analytical series are in perfect superposition, therefore only one plot is visible.

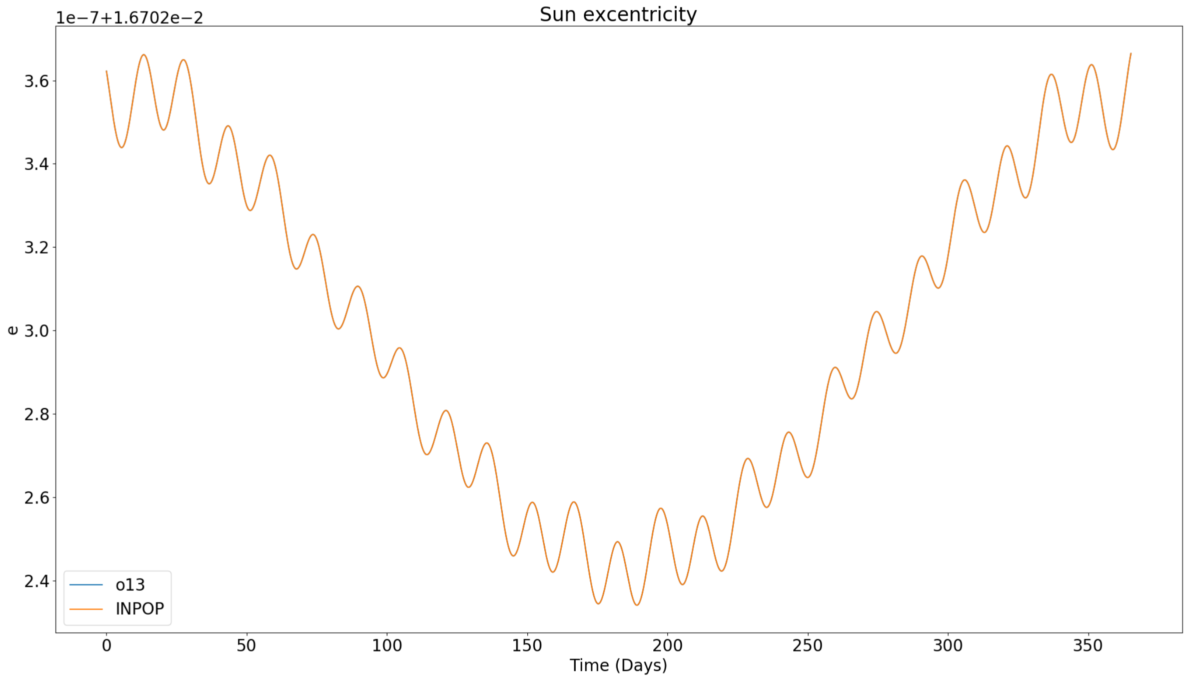


Figure 3: Excentricity of the (Earth-Moon barycenter)-Sun vector from J2000 to J2000+ 1 year

It could be interesting to plot the error between the numerical simulation and the analytical result over time for some orbital elements

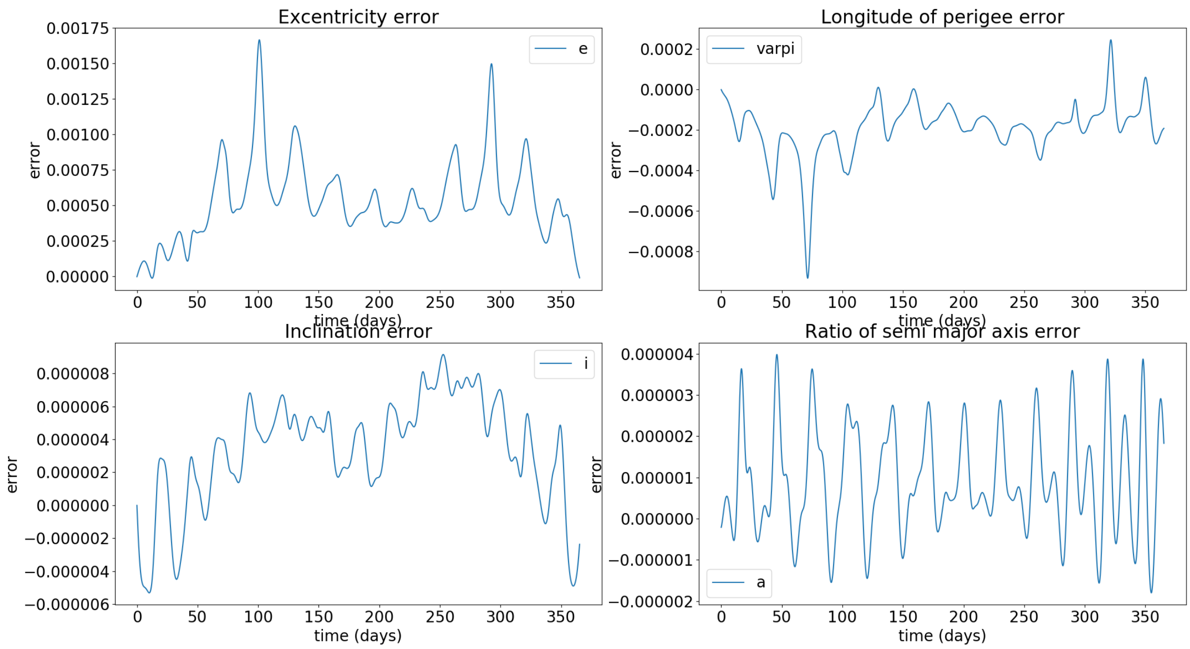


Figure 4: error from J2000 to J2000+ 1 year for some lunar orbital elements at order 13

We may notice that the error is greater for the excentricity than for other orbital elements. This is due to the fact that the excentricity is harder to approximate with series of the orbital elements than other orbital elements, yielding an important error when only few terms are taken.

This fact is better noticed if we plot the relative error over a time of 5000 years.

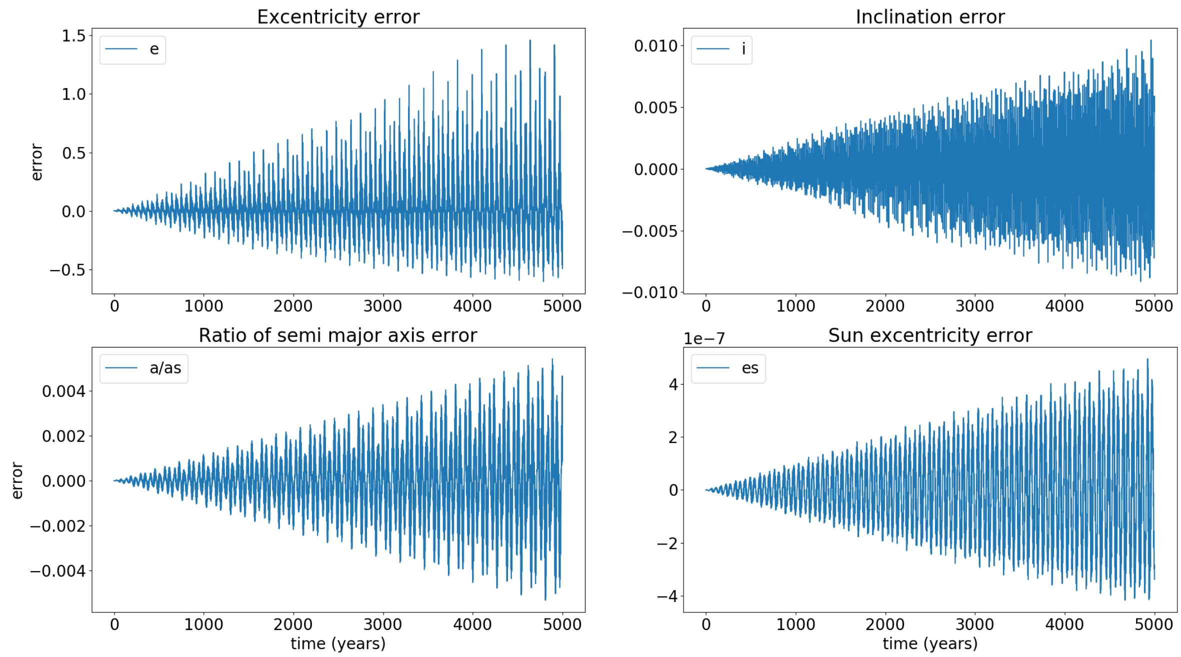


Figure 5: error from J2000 to J2000 + 5000 years for some lunar orbital elements at order 13

After only a few hundred years, the error on the Moon's excentricity reaches values as high as 10%. On the other hand, other orbital elements are far better approximated, as the relative error for the Moon's inclination, the ratio of semi-major axis and the Sun's excentricity don't exceed 1%, 0.5% and 5×10^{-7} after 5000 years, respectively.

It should be noted that this slow increase in the error is partly due to a slight difference in the frequencies between the numerical solution and the analytical one. Indeed, the frequencies in the analytical case are obtained from $\nu = \dfrac{\partial \mathcal{H}''}{\partial J''}$, and are thus necessarily wrong, as the non-integrable part of $\mathcal{H}''$ was neglected. Even though this fact prevents us from using these analytic results to make ephemeris over large times, it still enables us to use it to have trends over larger times.

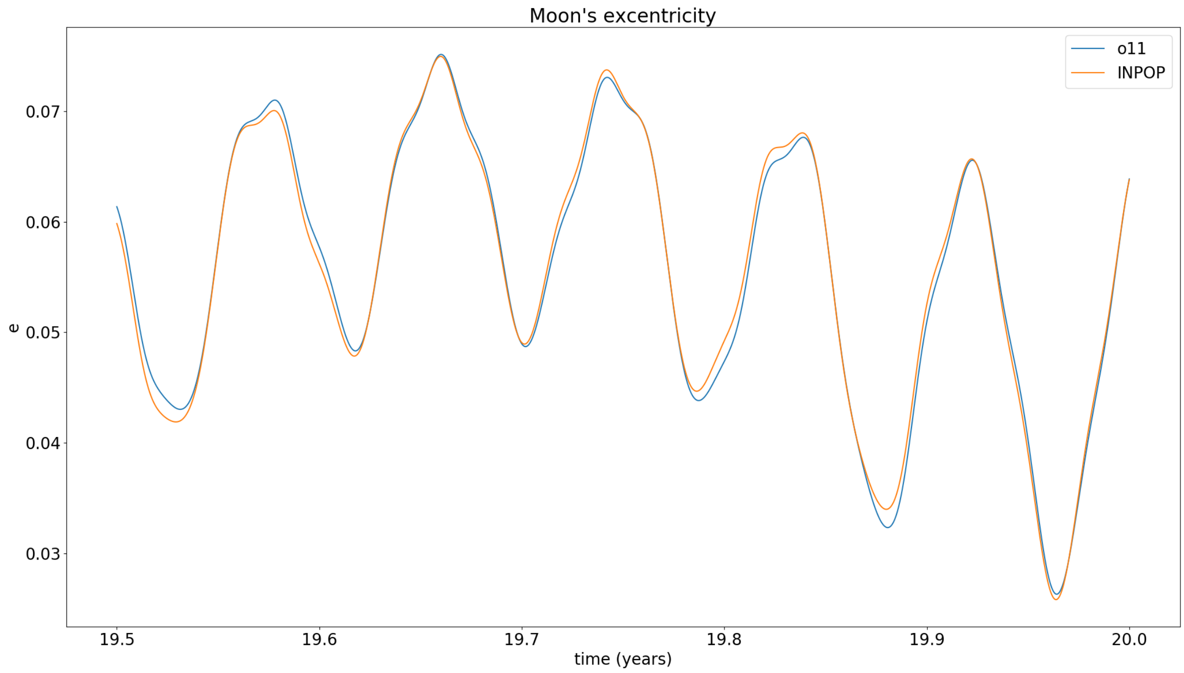


Figure 6: Moon excentricity from J2000 + 19.5 years to J2000 + 20 years at order 11

Even though the error is already substantial, we still have the tendency of the solution.

4 Tidal interactions and dissipation

In the unrestricted problem treated above, it is made the assumption that the three bodies, namely the Earth, the Sun and the Moon are point masses. With this simplification, all their mass is concentrated on the same point and they don't undergo differential gravitation from each others.

As can be seen in equation 40, the tidal potential undergone by a body $\mathbf{r}^*$ due to the tides it raises on a body $\mathbf{r}$ is proportional to $\dfrac{1}{|\mathbf{r}^* - \mathbf{r}|^6}$. As a consequence, the point-mass assumption yields no significant error as long as orbiting bodies are far from one another.

Nevertheless, in a realistic model, bodies in the neighbourhood of a planet raise tidal bulges on this planet. The departure from spherical symmetry of the planet modifies the potential this planets creates around it and the tidal-responsible bodies are subsequently affected by the tides.

In this section, we try and take into account these effects. For the sake of simplicity, we consider that only the Earth recovered its dimensions and that it is at hydrostatic equilibrium. There are two main reasons why a planet at hydrostatic equilibrium should depart from the spherical symmetry:

- Its own rotation that raises an equatorial bulge.

- Tides raised by other bodies.

In this study, both these effects will be considered.

4.1 Tidal potential

Let us consider a planet with barycenter O of mass m. The planet has no particular shape and is orbited by a point-mass satellite of mass m^* located in $\mathbf{r}^*$ in an inertial reference frame centered in O. Let P be any point inside the planet and $\mathbf{r} = \vec{OP}$. The gravitational potential between $\mathbf{r}$ and $\mathbf{r}^*$ is given by

$$V_{\text{grav}}(\mathbf{r}, \mathbf{r}^*) = -\frac{Gm^*}{|\mathbf{r}^* - \mathbf{r}|} = -\frac{Gm^*}{r^*} \sum_{l=0}^{+\infty} \left(\frac{r}{r^*}\right)^l P_l(\cos S^*) \tag{33}$$

where S^* is the angle between $\mathbf{r}$ and $\mathbf{r}^*$ and the P_l are the Legendre polynomials.

The terms $l = 0$ and $l = 1$ whose gradient complies with

$$-\nabla_{\mathbf{r}} \left[-\frac{Gm^*}{r^*} \left(1 + \frac{\mathbf{r}.\mathbf{r}^*}{r^{*2}}\right)\right] = \frac{Gm^* \mathbf{r}^*}{r^{*3}} \tag{34}$$

are responsible for the orbital motion of P around the satellite.

The terms $l = 2$ to ∞ are thus responsible for the reshaping of the planet by the satellite. The reshaping potential can be written as

$$V_{\text{resh}}(\mathbf{r}, \mathbf{r}^*) = -\frac{Gm^*}{r^*} \sum_{l=2}^{+\infty} \left(\frac{r}{r^*}\right)^l P_l(\cos S^*) = \sum_{l=2}^{+\infty} V^l_{\text{resh}}(\mathbf{r}, \mathbf{r}^*) \tag{35}$$

When the planet is reshaped by V_{resh}, its modified distribution of mass gives rise to a tidal potential V_{tide} that will influence any nearby body, located in $\mathbf{r}'$, that we will call undergoing body.

Studies on elastic deformations of planets performed by Kelvin, Darwin and Love suggest to take (Neron de Surgy and Laskar, 1997)

$$V_{\text{tide}}(|\mathbf{r}'| = R) = \sum_{l=2}^{+\infty} k_l V^l_{\text{resh}}(|\mathbf{r}| = R,\ \mathbf{r}^*) \tag{36}$$

where the k_l are the Love's numbers and R is the volumetric radius of the planet.

The general expression of V_{tide} is obtained thanks to the Poisson-like equation (Neron de Surgy and Laskar, 1997)

$$\Delta V_{\text{tide}}(\mathbf{r}') = 0 \text{ if } |\mathbf{r}'| \geq R \tag{37}$$

and the boundary condition given by equation 36

Let us use spherical coordinates, the general solution of equation 37 is hence given by

$$f(r,\theta,\phi) = \left\{ \begin{array}{c} r^l \\ r^{-1-l} \end{array} \right\} P_l^m\left(\cos\theta\right) \left\{ \begin{array}{c} e^{im\varphi} \\ e^{-im\varphi} \end{array} \right\} \qquad \begin{array}{c} l \in \mathbb{N} \\ -l \leq m \leq l \end{array}$$
$$\text{where } P_l^m\left(X\right) = \frac{1}{2^l l!}\left(1-X^2\right)^{m/2}\frac{d^{l+m}}{dX^{l+m}}\left(X^2-1\right)^l \tag{38}$$

Using the spherical harmonics defined as $Y_l^m\left(\theta,\varphi\right) = (-1)^m\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(\cos\theta)e^{im\varphi}$ we can now write V_{tide} as

$$V_{\text{tide}}(r',\theta,\varphi) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l} a_l^m r'^{-1-l} Y_l^m(\theta,\varphi)$$

Where the coefficients a_l^m can be obtained from equation 36

Some more calculation finally rises (Mignard, 1980)(Neron de Surgy and Laskar, 1997)

$$V_{\text{tide}}(\mathbf{r}') = -\frac{Gm^*}{R}\sum_{l=2}^{+\infty} k_l\left(\frac{R}{r^*}\right)^{l+1}\left(\frac{R}{r'}\right)^{l+1} P_l\left(\cos S\right) \tag{39}$$

where $\mathbf{r}^*$ is the position of the perturbing body and $\mathbf{r}'$ is the position of the undergoing body, that could be the same and $\cos S$ the angle between the two vectors.

In most problems, the greatness of r^* and r' with respect to R allows one to only consider the term $l = 2$, which finally yields the following tidal potential

$$V_{\text{tide}}(\mathbf{r}') = -k_2\frac{Gm^* R^5}{2r^{*3}r'^3}\left[3\cos^2 S - 1\right] \tag{40}$$

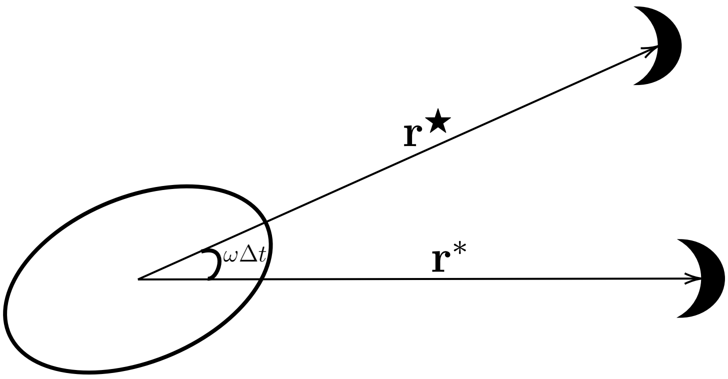
4.2 Tidal dissipation

In the particular case where the perturbing body, for example a satellite, is also the undergoing body, the tidal potential it feels can a priori be obtained from equation 40 by setting $\mathbf{r}^* = \mathbf{r}'$. Nevertheless, the purely elastic model we used means that the tidal bulge raised by the satellite is always aligned with $\mathbf{r}^*$, even though the planet rotates and the satellite moves aound the planet. As a consequence, no torque is exerted by the satellite on the tidal bulge and the spin of the planet is not affected. Moreover, by supposing that the planet is perfectly elastic, we suppose that no energy is dissipated inside its core by friction and thus, the total energy of the system remains the same.

In a more realistic model though, the planet is not elastic which means that energy is dissipated by heating up in the inside of the planet and the bulge is not perfectly aligned with the perturbing body. We would like to take this into account without having to thoroughly model the internal structure of the planet. We rely here on the work of (Mignard, 1980)

4.2.1 Inelasticity delay

The inelastic behaviour of the Earth is modeled by introducing a time delay Δt which corresponds to the time it takes to the planet to reach its equilibrium shape after a stress due to a perturbing body. Doing so, we reduce the modelisation of the internal structure of the planet to the sole parameter Δt. Let $\vec{\omega}$ be the rotation vector of the planet in any reference frame centered on the barycenter of the planet, $\mathbf{r}^*$ the position of the perturbing body, called the satellite, in that reference frame and $\mathbf{R}^*$ the position of the satellite in the frame that rotates with the planet. Let $\mathcal{R} = \mathcal{R}_{\vec{\omega}}(-\omega t)$ be the rotation matrix such that $\mathbf{r}^* = \mathcal{R}^{-1}\mathbf{R}^*$



To account for the fact that the planet is inelastic, we consider that the bulge raised at time t in the rotating frame isn't caused by $\mathbf{R}^*(t)$ but by $\mathbf{R}^*(t-\Delta t)$. This leads to the introduction of a fictional satellite $\bigstar$ such that $\mathbf{R}^{\bigstar}(t) = \mathbf{R}^*(t-\Delta t)$. We consider Δt to be negligible with respect to the rotational period of the planet, allowing us to perform order 1 expansions.

In the inertial reference frame, we can write

$$\mathbf{r}^{\bigstar} = \mathcal{R}^{-1}\left(\mathbf{R}^* - \Delta t\dot{\mathbf{R}}^*\right) = \mathbf{r}^* - \Delta t\left(\dot{\mathbf{r}}^* - \vec{\omega}\times\mathbf{r}^*\right) \tag{41}$$

To take into account dissipations and inelasticity in our model, we replace $\mathbf{r}^*$ by $\mathbf{r}^{\bigstar}$ in the tidal potential 40

Using the Andoyer variables defined by relations 49, we shall write

$$\mathbf{r}^{\bigstar} = \mathbf{r}^*(t-\Delta t) + (\lambda^* - \lambda')\,\mathbf{w}\times\mathbf{r}^*(t) \tag{42}$$

Where $\mathbf{w} = \dfrac{\vec{\omega}}{\omega}$ and $\lambda' = \lambda^* - \omega\Delta t$

At first order in $\omega\Delta t$, we can write

$$\mathbf{r}^{\bigstar} = \mathbf{r}^*(t-\Delta t) + (\lambda^* - \lambda')\,\mathbf{w}\times\mathbf{r}^*(t-\Delta t) \tag{43}$$

4.3 Interaction with the equatorial bulge

When a planet rotates, centrifugal forces tend to eject matter outside the axis of rotation of the planet, raising an equatorial bulge around the planet. As a consequence, bodies in the vicinity of the planet feel the potential created by this equatorial belt and their motion is affected, on the other hand, bodies who don't orbit in the equatorial plane of the planet exert a torque on the equatorial bulge and affect the spin of the planet.

4.3.1 Equatorial bulge potential

Let's assume that the planet is at hydrostatic equilibrium, thus, its angular momentum is collinear with its axis of highest inertia and the matrix of inertia of the planet can be written as

$$\mathcal{I} = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix}$$

A body outside the planet of mass m at position $\mathbf{r}^*$ will feel the potential

$$V_{\text{belt}}(\mathbf{r}^*) = -\int_{\text{planet}} \frac{G}{|\mathbf{r}^* - \mathbf{r}|} dm(\mathbf{r}) = -\frac{G}{r^*} \int_{\text{planet}} \sum_{n=0}^{+\infty} \left(\frac{r}{r^*}\right)^n P_l\left(\cos S\right) dm(\mathbf{r}) \tag{44}$$

where the P_l are the Legendre polynomials and $\cos S$ is the angle between $\mathbf{r}$ and $\mathbf{r}^*$

Noticing that ${}^t\mathbf{r}^*\mathcal{I}\mathbf{r}^* = \int_{\text{planet}} r^2 r^{*2} - (\mathbf{r}.\mathbf{r}^*)^2 \, dm(\mathbf{r})$ and $\int_{\text{planet}} r^2 \, dm(\mathbf{r}) = A + \frac{1}{2}C$ and truncating the Legendre expansion at order 2 we can write

$$V_{\text{belt}}(\mathbf{r}^*) = -\frac{Gm}{r^*} - \frac{G(C-A)}{2r^{*5}} \left[r^{*2} - 3(\mathbf{w}.\mathbf{r}^*)^2\right]$$

where $\mathbf{w} = \dfrac{\vec{\omega}}{\omega}$

The term $-\dfrac{Gm}{r^*}$ which corresponds to the point-mass gravitational potential should be excluded, yielding this final form for the potential caused by the equatorial belt (Boué and Laskar, 2006)

$$V_{\text{belt}}(\mathbf{r}^*) = -\frac{G(C-A)}{2r^{*5}} \left[r^{*2} - 3(\mathbf{w}.\mathbf{r}^*)^2\right] \tag{45}$$

4.3.2 Value of $C - A$

In the above expression, we would like to express $C - A$ as a function of the angular velocity ω of the planet. This can be achieved easily enough for an homogeneous planet when the hydrostatic equilibrium is supposed and when $\alpha = \dfrac{\omega^2}{\omega_c^2} \ll 1$ where $\omega_c = \sqrt{\dfrac{Gm_p}{R^3}}$ is the angular velocity for which centrifugal acceleration at the equator would compensate gravity if the planet remained spherical. The value of α for the Earth is around $\alpha = 0.00345$

In the rotating reference frame in spherical coordinates, the hydrostatic equilibrium reads

$$\frac{1}{\rho}\nabla P = -\omega_c^2 r\hat{e_r} + \omega^2 r\sin\theta\left(\sin\theta\hat{e_r} - \cos\theta\hat{e_\theta}\right) \tag{46}$$

yielding

$$P(r,\theta) = -\frac{1}{2}\rho\omega_c^2\left(1-\alpha\sin^2\theta\right)r^2 + K$$

where the constant K is obtained from the conservation of volume in such a way that the pressure at the surface is 0

One gets

$$K = \frac{1}{2}\rho\omega_c^2 R^2\left(1-\frac{2}{3}\alpha\right)$$

yielding the pressure everywhere inside the planet

$$P(r,\theta) = \frac{1}{2}\rho\omega_c^2 R^2\left(1-\frac{2}{3}\alpha-\gamma^2+\alpha\gamma^2\sin^2\theta\right) \qquad \gamma = \frac{r}{R} \tag{47}$$

The equation of the surface is given by the null isobar

$$\frac{r}{R} = 1 + \frac{1}{2}\alpha\left(\sin^2\theta - \frac{2}{3}\right)$$

The values of C and A are now given by

$$\begin{aligned}
C &= \frac{2}{5}m_p R^2 + \int_0^{2\pi}\int_0^{\pi}\int_R^{\frac{1}{2}\alpha R\left(\sin^2\theta-\frac{2}{3}\right)} \rho r^4\sin^3\theta drd\theta d\phi \\
A &= \frac{2}{5}m_p R^2 + \int_0^{2\pi}\int_0^{\pi}\int_R^{\frac{1}{2}\alpha R\left(\sin^2\theta-\frac{2}{3}\right)} \rho r^4\sin\theta\cos^2\theta drd\theta d\phi
\end{aligned}$$

And we end up with

$$\begin{aligned}
& C = C_{\text{sp}}\left(1+\frac{1}{3}\alpha\right) \qquad A = C_{\text{sp}}\left(1-\frac{1}{6}\alpha\right) \\
& C_{\text{sp}} = \frac{2}{5}m_p R^2 \\
& C - A = \frac{1}{5}m_p R^2\alpha
\end{aligned} \tag{48}$$

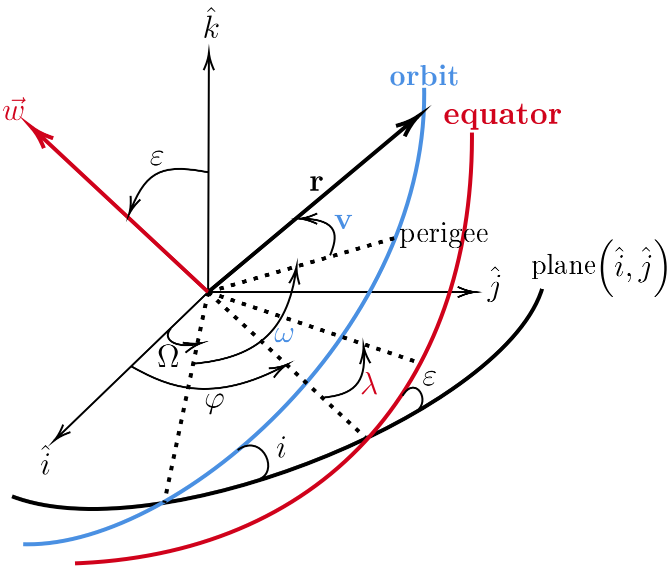
4.4 Establishment of $\mathcal{H}$

In this section, we will sum up the results of the previous sections to get an expression of the hamiltonian of the Star-Planet-Satellite problem in the case where the planet can undergo tides.

The canonical Andoyer variables are used to described the rotation of the planet and are defined as

$$\begin{array}{c|c} \text{Action} & \text{Angle} \\ \Lambda = C\omega & \lambda \\ X = \Lambda\cos\varepsilon & \varphi \end{array} \tag{49}$$

Where λ is the rotation angle of the planet and φ its precession angle.



4.4.1 In Jacobi and Andoyer variables

Using these variables, the kinetic energy of rotation of the planet is written as

$$T = \frac{\Lambda^2}{2C} = \frac{5\Lambda^2}{4m_pR^2}\left(1 - \frac{25\Lambda^2}{12Gm_p^3R}\right) \tag{50}$$

If we describe the positions of the bodies with Jacobi coordinates defined by equation 23, the Planet-Star vector is written $\delta\mathbf{r} + \mathbf{r_s}$ where $\delta = \dfrac{m_{\text{sat}}}{m_{\text{sat}} + m_{\text{pla}}}$, $\mathbf{r}$ is the Planet-Satellite vector and $\mathbf{r_s}$ is the (Planet-Satellite barycenter)-Star vector. From sections 4.1, 4.2 and 4.3, we finally write the hamiltonian of the problem under the form

$$\mathcal{H} = \mathcal{H}_{\text{point-mass}} + \frac{5\Lambda^2}{4m_pR^2}\left(1 - \frac{25\Lambda^2}{12Gm_p^3R}\right) + \mathcal{H}_{\text{belt}} + \mathcal{H}_{\text{tide}} \tag{51}$$

where $\mathcal{H}_{\text{point-mass}}$ is obtained in section 3.3, $\mathcal{H}_{\text{belt}}$ is given by

$$\mathcal{H}_{\text{belt}} = -\frac{5\Lambda^2Rm}{8m_p^2r^3}\left[1 - 3\left(\frac{\mathbf{r}.\mathbf{w}}{r}\right)^2\right] - \frac{5\Lambda^2Rm_s}{8m_p^2\left|\delta\mathbf{r} + \mathbf{r_s}\right|^3}\left[1 - 3\left(\frac{(\delta\mathbf{r} + \mathbf{r_s})\,.\mathbf{w}}{\left|\delta\mathbf{r} + \mathbf{r_s}\right|}\right)^2\right] \tag{52}$$

and $\mathcal{H}_{\text{tide}}$ reads

$$
\begin{aligned}
&\mathcal{H}_{\text{tide}} = \mathcal{H}_{\text{sat/sat}} + \mathcal{H}_{\text{sat/star}} + \mathcal{H}_{\text{star/sat}} + \mathcal{H}_{\text{star/star}} \\
&\mathcal{H}_{\text{sat/sat}} = -k_2 \frac{Gm^2R^5}{2r^{\star 5}r^5}\left[3\left(\mathbf{r}.\mathbf{r}^{\star}\right)^2 - r^2 r^{\star 2}\right] \\
&\mathcal{H}_{\text{sat/star}} = -k_2 \frac{Gmm_sR^5}{2\left|\delta\mathbf{r}+\mathbf{r_s^{\star}}\right|^5 r^5}\left[3\left(\mathbf{r}.\left(\delta\mathbf{r}+\mathbf{r_s^{\star}}\right)\right)^2 - r^2\left|\delta\mathbf{r}+\mathbf{r_s^{\star}}\right|^2\right] \\
&\mathcal{H}_{\text{star/sat}} = -k_2 \frac{Gmm_sR^5}{2\left|\delta\mathbf{r}+\mathbf{r_s}\right|^5 r^{\star 5}}\left[3\left(\mathbf{r}^{\star}.\left(\delta\mathbf{r}+\mathbf{r_s}\right)\right)^2 - r^{\star 2}\left|\delta\mathbf{r}+\mathbf{r_s}\right|^2\right] \\
&\mathcal{H}_{\text{star/star}} = -k_2 \frac{Gm_s^2R^5}{2\left|\delta\mathbf{r}+\mathbf{r_s}\right|^5\left|\delta\mathbf{r}+\mathbf{r_s^{\star}}\right|^5}\left[3\left(\left(\delta\mathbf{r}+\mathbf{r_s^{\star}}\right).\left(\delta\mathbf{r}+\mathbf{r_s}\right)\right)^2 - \left|\delta\mathbf{r}+\mathbf{r_s^{\star}}\right|^2\left|\delta\mathbf{r}+\mathbf{r_s}\right|^2\right]
\end{aligned}
\tag{53}
$$

And, in accordance with notations from section 3.1, s subscript stands for the Star and no subscript stands for the Satellite. Unlike section 3.1, we need to refer to the Planet and we do so with a p subscript.

4.4.2 In Poincaré variables

We now would like to express our hamiltonian in function of the orbital elements and then the Poincaré variables and the quantities defined in relations 29.

Let's associate orbital elements $a', e', i', \Omega', \varpi', M'$ and $a'_s, e'_s, i'_s, \Omega'_s, \varpi'_s, M'_s$ to $\mathbf{r}(t-\Delta t)$ and $\mathbf{r_s}(t-\Delta t)$ respectively. From equations 17 and 18, we are able to express terms of the form $\mathbf{r_1}.\mathbf{r_2}$ where $\mathbf{r_1}$ and $\mathbf{r_2} \in \{\mathbf{r}(t), \mathbf{r_s}(t), \mathbf{r}(t-\Delta t), \mathbf{r_s}(t-\Delta t)\}$, and from relations 54, we can express terms of the form $\mathbf{w}.(\mathbf{r_1}\times\mathbf{r_2})$. Finally, we can express $\mathcal{H}_{\text{tide}}$ as a function of λ, λ', the primed orbital elements and the unprimed ones.

We already expressed the term of kinetic energy of rotation as a function of Λ so nothing left is to be done.

Expressing $\mathcal{H}_{\text{point-mass}}$ also uses equations 17 and 18 and is detailled in section 3.3.2

To express $\mathcal{H}_{\text{belt}}$, we need to compute terms of the form $\mathbf{r}.\mathbf{w}$ and $\mathbf{r_s}.\mathbf{w}$. From the figure at the beginning of section 4.4, we can write

$$
\begin{aligned}
&\mathbf{r} = r\mathcal{R}_3(\Omega)\mathcal{R}_1(i)\mathcal{R}_3(\omega)\mathcal{R}_3(v)\begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \mathbf{w} = \mathcal{R}_3(\varphi)\mathcal{R}_1(\varepsilon)\begin{pmatrix}0\\0\\1\end{pmatrix} \\
&\mathcal{R}_3(\theta) = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix} \quad \mathcal{R}_1(\theta) = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}
\end{aligned}
\tag{54}
$$

We end up with

$$
\frac{\mathbf{r}}{r}.\mathbf{w} = \sin\varepsilon\cos(\omega+v)\sin(\varphi-\Omega) - \sin\varepsilon\cos i\sin(\omega+v)\cos(\varphi-\Omega) + \cos\varepsilon\sin i\sin(\omega+v) \tag{55}
$$

4.5 Establishment of the equations of motion

As tidal phenomena are not conservative, no potential exists from which tidal forces derive. This means that the hamiltonian formalism doesn't apply when tides are present and we shouldn't have been able to end up to the expression of a canonical hamiltonian in previous section. In fact, our way of modeling dissipation in section 4.2 implies the introduction of a fictional perturbing body $\bigstar$ whose orbital elements are related to those of the non fictional perturbing body $*$. If we express the orbital elements of $\bigstar$ as a function of those of $*$ in our hamiltonian $\mathcal{H}$, we won't be able to derive the equations of motion from the Hamilton-Jacobi equations 2 and our hamiltonian won't be canonical anymore.

Indeed, when deriving the hamiltonian with respect to $\mathbf{r}^*$ in order to obtain the time derivative of $\tilde{\mathbf{r}}^*$, we only want $\mathbf{r}^*$ to infinitesimally move, and not $\mathbf{r}^\bigstar$, as this latter was only an artifact introduced to take into account the inelasticity of the Earth. That is why, when deriving the hamiltonian with respect to $\mathbf{r}^*$ to obtain the equations of motion, $\mathbf{r}^\bigstar$ should be viewed as constant. Thereafter only should it be replaced by its expression in terms of $\mathbf{r}^*$

Nevertheless, in order to use the Lie formalism exposed in section 1, we need to get rid of all occurrences of $\bigstar$ beforehand in our hamiltonian as $\bigstar$ doesn't really exist, and it means replacing it by its $*$ dependent expression. As we can't do so, section 1 doesn't apply here.

What we will do instead is the following procedure:

- step 1 | Obtain the hamiltonian of the problem written in term of Poincaré variables using previous results of section 4
- step 2 | Write the hamiltonian in terms of the quantities defined in 29 using relations 30
- step 3 | Associate fictional primed orbital elements to $\mathbf{r}(t - \Delta t)$ and $\mathbf{r_s}(t - \Delta t)$
- step 4 | Derive the equation of motion using equation 2 and relations 31 and 32. The primed variables must be kept constant during the derivation.
- step 5 | Perform the change of variable defined by the generators W' and W'' established in section 3.3.4 on the hamiltonian.

Once step 5 is completed, $\mathcal{H}$ and all its variables should be double-primed to be consistent with notations from section 3 but, to avoid ambiguity with the fictional primed orbital elements, we will drop the double-prime.

- step 6 | Get rid of the fictional primed variables by expressing them in term of the unprimed ones
- step 7 | Integrate the equations of motion using results from section 4.6

Note that step 6 appears after step 4, accordingly to what was said before.
Step 6 is performed like so

$$\begin{aligned} a' &= a \\ e' &= e \\ i' &= i \\ l' &= l - n\Delta t \\ g' &= g \\ h' &= h \\ \lambda' &= \lambda - \omega\Delta t \end{aligned} \qquad \text{Which traduces into} \qquad \begin{aligned} x' &= x \\ y' &= y \\ \eta' &= \eta \\ l' &= l - n\Delta t \\ g' &= g \\ h' &= h \\ \lambda' &= \lambda - \omega\Delta t \end{aligned}$$

Where λ' is defined in equation 43

What prevents us from using the hamiltonian formalism for this part is the presence of dissipative effects in the hamiltonian $\mathcal{H}_{\text{tide}}$. Nevertheless, we could still apply the Lie formalism on the Hamiltonian $\mathcal{H}_{\text{point-mass}} + \frac{5\Lambda^2}{4m_pR^2}\left(1 - \frac{25\Lambda^2}{12Gm_p^3R}\right) + \mathcal{H}_{\text{belt}}$ as this hamiltonian doesn't traduce any loss in energy. Thereafter only could we apply the 7 previous steps. It is a priori not compulsory to do so.

4.6 Integration of the equations of motion

Once all 7 steps of section 4.5 have been completed, we end up with a differential equation of the form

$$\dot{x} = F(x) \tag{56}$$

Where $x = {}^t(L, G, H, L_s, G_s, H_s, \Lambda, X, l, g, h, l_s, g_s, h_s, \lambda, \varphi) = \begin{pmatrix} Y \\ y \end{pmatrix} \in \mathbb{R}^{16}$ and F is the vector field defined by $F\begin{pmatrix} Y \\ y \end{pmatrix} = {}^t\left(-\frac{\partial \mathcal{H}}{\partial y}, \frac{\partial \mathcal{H}}{\partial Y}\right)$ with the order step 5 $\to$ step 6 having been respected.

F can be written $F = F_0 + F_1$ with

$$\begin{aligned} F_0(x) &= {}^t\left(-\frac{\partial \mathcal{H}_0}{\partial y}, \frac{\partial \mathcal{H}_0}{\partial Y}\right) \qquad & F_1(x) &= {}^t\left(-\frac{\partial \mathcal{H}_1}{\partial y}, \frac{\partial \mathcal{H}_1}{\partial Y}\right) \\ \mathcal{H}_0 &= \mathcal{H}_{\text{point-mass}} \qquad & \mathcal{H}_1 &= \mathcal{H} - \mathcal{H}_0 \end{aligned}$$

Thanks to step 4, we have

$$F_0(x) = {}^t(0, 0, 0, 0, 0, 0, 0, 0, \nu_L, \nu_G, \nu_H, \nu_{L_s}, \nu_{G_s}, \nu_{H_s}, 0, 0) = \text{Cte}$$

And equation 56 is extremely easily solved when F_1 is neglected. On the contrary, F_1 can have whatever form in the coordinate system we went into during step 4, and we won't a priori be able to analytically solve equation 56 without neglecting F_1

4.6.1 Change of variable

We would like to perform a change of coordinates $x' = \Phi(x)$, close from the identity, such that in this new coordinate system, we have

$$\dot{x}' = F_0(x')$$

That we can analytically solve.

We write

$$\Phi = I_d + Z \tag{57}$$

Where Z is a vector field which complies with $Z \ll I_d$, such that we have, at first order, $\Phi^{-1} = I_d - Z + \mathcal{O}\left(Z^2\right)$

Performing first order expansions, equation 56 becomes

$$\dot{x}' = F_0(x') - dF_0(x')Z(x') + F_1(x') + dZ(x')F_0(x') \tag{58}$$

To reach our goal, we set

$$F_1(Y', y') = dF_0(Y')Z(Y', y') - dZ(Y', y')F_0(Y') \tag{59}$$

Which is the vectorial equivalent of the scalar cohomological equation 5

The Jacobian of F_0 and Z satisfy

$$dF_0(Y') = \left(\begin{array}{c|c} 0_n & 0_n \\ \hline * & 0_n \end{array}\right) \text{ and } dZ(x') = \left(\begin{array}{c|c} \left(\dfrac{\partial Z_i}{\partial Y_j}\right)_{1\leq i,j\leq n} & \left(\dfrac{\partial Z_i}{\partial y_j}\right)_{1\leq i,j\leq n} \\ \hline \left(\dfrac{\partial Z_{i+n}}{\partial Y_j}\right)_{1\leq i,j\leq n} & \left(\dfrac{\partial Z_{i+n}}{\partial y_j}\right)_{1\leq i,j\leq n} \end{array}\right)$$

And the vectorial cohomological equation 59 reduces to

$$F_1(Y', y') = \begin{pmatrix} 0 \\ | \\ 0 \\ f_1(Y')(Z_1, ..., Z_n) \\ | \\ f_n(Y')(Z_1, ..., Z_n) \end{pmatrix} - \begin{pmatrix} \displaystyle\sum_{k=1}^{n} \nu_k \frac{\partial Z_1}{\partial y_k} \\ | \\ \displaystyle\sum_{k=1}^{n} \nu_k \frac{\partial Z_{2n}}{\partial y_k} \end{pmatrix} \tag{60}$$

The vectorial cohomological equation is thus solved by solving $2n$ independent scalar cohomological equations, n being 8 in our case.

Results from subsection 1.2.2 can be used here. Let us write

$$\begin{cases} -F_{1_i} = \psi_0 + \displaystyle\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \psi_k e^{ik.y'} & \text{if } 1 \leq i \leq n \\ f_{i-n} - F_{1_i} = \psi_0 + \displaystyle\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \psi_k e^{ik.y'} & \text{if } n+1 \leq i \leq 2n \end{cases}$$

And

$$Z_i = z_0 + \sum_{k\in\mathbb{Z}^n\backslash\{0\}} z_k e^{ik.y'}$$

We obtain

$$z_0 = \psi_0$$
$$z_k = \frac{\psi_k}{i\,k.y'}$$

4.6.2 Going back into the old variables

In the new primed variables, the equation of motion read

$$\dot{x}' = F_0(x')$$

And their solutions are given by

$$Y' = Y_0' \quad \text{and} \quad y' = y_0' + F_0(Y_0')t$$

The expressions of Y and y as a function of time are obtained by writing $x = x' - Z(x')$

5 Conclusion

Throughout this study, we have analytically solved the Three-Body problem, going toward a more and more general problem as we first considered the restricted problem, then we regarded the unrestricted one where the satellite had recovered its mass and finally, whe took into account tidal and dissipative effects, where the planet had recovered its dimension. Due to frequency shifts, our analytic model was able to produce ephemeris for a certain amount of time only, but could give us the trend of the orbital elements for far larger times, something a numerical integration can't do as it slowly diverges over time.

As this study produces plots of the results for the unrestricted problem, as well as explicit expressions of some of the series manipulated by *TRIP* in appendix, it doesn't do so for the tidal effects. Indeed, only the theoretical work has been done for this part, as we still hadn't time to finish the code at the time of this report. Nevertheless, all results needed to do so are in this report and we hope to be able to produce plots for the oral defense on June the 25th 2019, or at least direct applications of the theory.

Even though it was dedicated to study the Sun-Earth-Moon system, this model could be adapted to any reasonable Star-Planet-Satellite system, provided the truncation rules are defined consequently.

The Lie formalism, used throughout this study, is a very powerful tool to deal with hamiltonian systems, and can even be adapted in cases where the system is almost hamiltonian, as was done in section 4. Nevertheless, the method asks huge amounts of computation that can only be performed by an algebraic manipulator and gives rises to tremendous analytic expressions. Moreover, even if they were not discussed in this study, some issues can arise from the cohomological equation in the case where resonances exist in the system, and the method couldn't be adapted as such to a system like Sun-Jupiter-Trojans, for example. Despite its flaws, the Lie formalism stays one of the only way to get analytic expressions of highly non-integrable problems such as a three-body problem submitted to tidal phenomena.

Appendices

A Development of $\mathcal{H}_{\mathbf{pendulum}}$ up to $N = 5$

Here are the results given by *TRIP*, under the form they were output, with $N = 5$. Note that $a = \omega^2$.

$$\mathcal{H}' =$$

```
5/64*a**4*L'**-6
+ 1/4*a**2*L'**-2
+ 1/2*L'**2
```

$$W =$$

```
- 1/384*a**5*L'**-9        *sin(5*l')
- 119/1536*a**5*L'**-9     *sin(3*l')
- 239/1536*a**5*L'**-9     *sin(l')
- 7/768*a**4*L'**-7        *sin(4*l')
- 13/96*a**4*L'**-7        *sin(2*l')
- 1/32*a**3*L'**-5         *sin(3*l')
- 5/32*a**3*L'**-5         *sin(l')
- 1/8*a**2*L'**-3          *sin(2*l')
- 1*a*L'**-1               *sin(l')
```

$$\nu =$$

```
- 15/32*a**4*L0'**-7\\
- 1/2*a**2*L0'**-3\\
+ 1*L0'\\
```

$$l_0' =$$

```
- 193/640*a**5*L0**-10  *sin(5*l0)
- 591/128*a**5*L0**-10  *sin(3*l0)
- 99/16*a**5*L0**-10    *sin(l0)
- 93/256*a**4*L0**-8    *sin(4*l0)
- 47/16*a**4*L0**-8     *sin(2*l0)
- 11/24*a**3*L0**-6     *sin(3*l0)
- 11/8*a**3*L0**-6      *sin(l0)
- 5/8*a**2*L0**-4       *sin(2*l0)
- 1*a*L0**-2            *sin(l0)
+ 1*l0
```

$$L_0' =$$

```
- 7/128*a**5*L0**-9     *cos(5*l0)
- 175/128*a**5*L0**-9   *cos(3*l0)
- 175/32*a**5*L0**-9    *cos(l0)
- 5/64*a**4*L0**-7      *cos(4*l0)
- 5/4*a**4*L0**-7       *cos(2*l0)
- 45/32*a**4*L0**-7
- 1/8*a**3*L0**-5       *cos(3*l0)
- 9/8*a**3*L0**-5       *cos(l0)
- 1/4*a**2*L0**-3       *cos(2*l0)
- 1/2*a**2*L0**-3
- 1*a*L0**-1            *cos(l0)
+ 1*L0
```

$$l(t) =$$

```
1/1280*L0'**-10*a**5     *sin(5*l0' + 5*nu*t)
+ 1/256*L0'**-8*a**4     *sin(4*l0' + 4*nu*t)
+ 1/48*L0'**-6*a**3      *sin(3*l0' + 3*nu*t)
+ 3/64*L0'**-10*a**5     *sin(3*l0' + 3*nu*t)
+ 1/8*L0'**-4*a**2       *sin(2*l0' + 2*nu*t)
+ 3/16*L0'**-8*a**4      *sin(2*l0' + 2*nu*t)
+ 1*L0'**-2*a            *sin(l0' + nu*t)
+ 11/16*L0'**-6*a**3     *sin(l0' + nu*t)
+ 247/256*L0'**-10*a**5  *sin(l0' + nu*t)
+ 1*t*nu
+ 1*l0'
```

$$L(t) =$$

```
1/256*L0'**-9*a**5     *cos(5*l0' + 5*nu*t)
+ 1/64*L0'**-7*a**4    *cos(4*l0' + 4*nu*t)
+ 1/16*L0'**-5*a**3    *cos(3*l0' + 3*nu*t)
+ 7/64*L0'**-9*a**5    *cos(3*l0' + 3*nu*t)
+ 1/4*L0'**-3*a**2     *cos(2*l0' + 2*nu*t)
+ 1/4*L0'**-7*a**4     *cos(2*l0' + 2*nu*t)
+ 1*L0'**-1*a          *cos(l0' + nu*t)
+ 3/16*L0'**-5*a**3    *cos(l0' + nu*t)
+ 39/256*L0'**-9*a**5*cos(l0' + nu*t)
+ 1*L0'
- 1/2*L0'**-3*a**2
- 15/32*L0'**-7*a**4
```

B *TRIP* code to obtain $\mathcal{H}_1$

Here is shown the source code that enabled me to express the hamiltonian $\mathcal{H}_1$ as a function of the quantities defined in relations 29 and the variables of Poincaré.

```
macro main{
  _echo off$
  _modenum=NUMRATMP$ // To perform exact computation
  _affdist=4$  // To display series in terms of sinus and cosinus
  _cleanflag=0$
  _mode = POLP$
  1*eps$ // To sort series by order

  delpow=1$  //The order of the different small quantities
  xipow=1$
  etapow=2$
  xpow=1$
  ypow=1$
  xspow=1$
  kappow=4$
  yspow=5$

  %init_angle$
  N=13$
  tel=({eps,N})$
  tro=({rsurrs,N})$
  tr=(tro,tel)$
  usetronc(tel)$
  %cosS$
  cosS=%simplifysq2[cosS]$
  %H1$
  H1;
  saveF(H1)$

}$

macro init_angle{
  ll=expi(l,1,0)$      //complex mean anomaly of the moon
  gg=expi(g,1,0)$      //complex argument of the periapsis of the moon
  hh=expi(h,1,0)$      //complex longitude of the ascending node of the moon
  lls=expi(ls,1,0)$                                    //d(ls)/dt=nuk
  ggs=expi(gs,1,0)$    //same but for the sun
  hhs=expi(hs,1,0)$
}$

macro simplifysq2[ser]{
  private nmin,nmax,a,terme,inutile,parite,serie,n$
  serie=ser$
  nmin=puismin(serie,sq2)$
  nmax=puismax(serie,sq2)$
  n=max(abs(nmax),abs(nmin))$
  for a=0 to n step 2{
    terme=coef_ext(serie,(sq2,a))$
```

```
    if(terme != 0) then {
      serie=serie-sq2**a*terme+2**(a/2)*terme$
    }$

    terme=coef_ext(serie,(sq2,-a))$
    if(terme != 0) then {
      serie=serie-sq2**-a*terme+2**(-a/2)*terme$
    }$
  }$
  for a=1 to n step 2{
    terme=coef_ext(serie,(sq2,a))$
    if(terme != 0) then {
      serie=serie-sq2**a*terme+2**((a-1)/2)*terme*sq2$
    }$

    terme=coef_ext(serie,(sq2,-a))$
    if(terme != 0) then {
      serie=serie-sq2**-a*terme+2**((-a-1)/2)*terme*sq2$
    }$
  }$
  return serie$
}$

macro cosS{

  private si,sis,subste,substes$
  efftronc$
  Expiv(N+1)$
  vv=subst(_Expiv,X,ll*gg)$
  vvs=subst(_Expiv,X,lls*ggs)$
  usetronc(tel)$

  subste=sq2*x*eps**(xpow)*(1-x**2*eps**(2*xpow)/2)**(1/2)$
  substes=sq2*xs*eps**(xspow)*(1-xs**2*eps**(2*xspow)/2)**(1/2)$
  vv=subst(vv,e,subste)$
  vvs=subst(vvs,e,substes)$

  si=(1/(1-x**2*eps**(2*xpow)))**(1/2)$ // si is gamma=sin(i/2)
  sis=(1/(1-xs**2*eps**(2*xspow)))**(1/2)$
  si=si*y*eps**(ypow)/sq2$
  sis=sis*ys*eps**(yspow)/sq2$

  SS=gg**-1*ggs**-1*vv*vvs*(si*hh*(1-sis**2)**(1/2)-
  sis*hhs*(1-si**2)**(1/2))**2+ggs*gg**-1*vv*vvs**-1
  *(((1-si**2)*(1-sis**2))**(1/2)+si*sis*hh*hhs**-1)**2$

  cosS=real(SS)$
}$

macro H1{
  private rsa,asrs,ro,J,subste,substes$
  efftronc$
```

```
  RsA(N+1)$
  AsR(N+1)$
  rsa=subst(_RsA,X,ll*gg)$
  asrs=subst(_AsR,X,lls*ggs)$
  usetronc(tel)$
  subste=sq2*x*eps**(xpow)*(1-x**2*eps**(2*xpow)/2)**(1/2)$
  substes=sq2*xs*eps**(xspow)*(1-xs**2*eps**(2*xspow)/2)**(1/2)$
  rsa=subst(rsa,e,subste)$
  asrs=subst(asrs,e,substes)$
  rsa=%simplifysq2[rsa]$
  asrs=%simplifysq2[asrs]$


  usetronc(tr)$
  rsa*asrs$
  ro=rsa*asrs*kap**-1*xi**-2*eta**4*eps**(4*etapow-2*xipow-kappow)$

  J=1-(1-del*eps**(delpow))*(1+del**2*eps**(2*delpow)*rsurrs**2+
  2*del*eps**(delpow)*rsurrs*s)**(-1/2)-del*eps**(delpow)*
  (1+(1-del*eps**(delpow))**2*rsurrs**2-2*(1-del*eps**(delpow))
  *rsurrs*s)**(-1/2)$

  J=subst(J,s,cosS)$
  J=subst(J,rsurrs,ro)$
  H1=asrs*J$
  H1=%simplifysq2[H1]$
}$
```

C Explicit expression of $\mathcal{H}_1$ in the three-body problem

Here is given the expression of the perturbative part of the hamiltonian of the three body problem as a function of the Poincaré variables and the quantities defines in 29, such as output by *TRIP*. Note that ε (eps) is a fictional variable whose power indicates the order of the term while *sq2* and *del* are $\sqrt{2}$ and δ respectively. Terms up to the 8$^{\text{th}}$ order are displayed

We have

$$\frac{L_s^2\xi^4\kappa^2}{\mu_s^2\beta_s^3\eta^8}\mathcal{H}_1 =$$

```
- 3/4*del*eps**5                              *cos( - 2*ls + 2*l)
- 1/4*del*eps**5
- 3/4*del*sq2*x*eps**6                        *cos( - 2*ls + g + 3*l)
+ 3/8*del*sq2*xs*eps**6                       *cos(   gs - ls + 2*l)
+ 3/4*del**2*eps**6                           *cos( - 2*ls + 2*l)
- 21/8*del*sq2*xs*eps**6                      *cos( - gs - 3*ls + 2*l)
+ 1/2*del*sq2*x*eps**6                        *cos(   g + l)
+ 9/4*del*sq2*x*eps**6                        *cos( - 2*ls - g + l)
- 3/4*del*sq2*xs*eps**6                       *cos(   gs + ls)
+ 1/4*del**2*eps**6
```

```
- 3/2*del*x**2*eps**7                              *cos( - 2*ls + 2*g + 4*l)
+ 3/4*del*xs*x*eps**7                              *cos(   gs - ls + g + 3*l)
+ 3/4*del**2*sq2*x*eps**7                          *cos( - 2*ls + g + 3*l)
- 21/4*del*xs*x*eps**7                             *cos( - gs - 3*ls + g + 3*l)
- 5/8*del*xi**-2*kap**-1*eta**4*eps**7             *cos( - 3*ls + 3*l)
+ 1/4*del*x**2*eps**7                              *cos(   2*g + 2*l)
- 3/4*del*y**2*eps**7                              *cos(   2*h + 2*l)
- 3/8*del**2*sq2*xs*eps**7                         *cos(   gs - ls + 2*l)
+ 15/4*del*xs**2*eps**7                            *cos( - 2*ls + 2*l)
+ 3/4*del*y**2*eps**7                              *cos( - 2*ls + 2*l)
+ 15/4*del*x**2*eps**7                             *cos( - 2*ls + 2*l)
+ 21/8*del**2*sq2*xs*eps**7                        *cos( - gs - 3*ls + 2*l)
- 51/4*del*xs**2*eps**7                            *cos( - 2*gs - 4*ls + 2*l)
+ 3/2*del*xs*x*eps**7                              *cos(   gs + ls + g + l)
- 1/2*del**2*sq2*x*eps**7                          *cos(   g + l)
+ 3/2*del*xs*x*eps**7                              *cos( - gs - ls + g + l)
- 3/8*del*xi**-2*kap**-1*eta**4*eps**7             *cos( - ls + l)
- 9/4*del*xs*x*eps**7                              *cos(   gs - ls - g + l)
- 9/4*del**2*sq2*x*eps**7                          *cos( - 2*ls - g + l)
+ 63/4*del*xs*x*eps**7                             *cos( - gs - 3*ls - g + l)
- 15/4*del*x**2*eps**7                             *cos(   2*ls + 2*g)
- 3/4*del*y**2*eps**7                              *cos(   2*ls + 2*h)
- 9/4*del*xs**2*eps**7                             *cos(   2*gs + 2*ls)
+ 3/4*del**2*sq2*xs*eps**7                         *cos(   gs + ls)
- 3/4*del*xs**2*eps**7
+ 3/4*del*y**2*eps**7
- 3/4*del*x**2*eps**7
- 25/16*del*sq2*x**3*eps**8                        *cos( - 2*ls + 3*g + 5*l)
+ 3/4*del*sq2*xs*x**2*eps**8                       *cos(   gs - ls + 2*g + 4*l)
+ 3/2*del**2*x**2*eps**8                           *cos( - 2*ls + 2*g + 4*l)
- 21/4*del*sq2*xs*x**2*eps**8                      *cos( - gs - 3*ls + 2*g + 4*l)
- 15/16*del*xi**-2*kap**-1*sq2*eta**4*x*eps**8*cos( - 3*ls + g + 4*l)
+ 1/8*del*sq2*x**3*eps**8                          *cos(   3*g + 3*l)
- 3/4*del*sq2*y**2*x*eps**8                        *cos(   2*h + g + 3*l)
- 3/4*del**2*xs*x*eps**8                           *cos(   gs - ls + g + 3*l)
+ 15/4*del*sq2*xs**2*x*eps**8                      *cos( - 2*ls + g + 3*l)
+ 3/4*del*sq2*y**2*x*eps**8                        *cos( - 2*ls + g + 3*l)
+ 15/4*del*sq2*x**3*eps**8                         *cos( - 2*ls + g + 3*l)
+ 21/4*del**2*xs*x*eps**8                          *cos( - gs - 3*ls + g + 3*l)
- 51/4*del*sq2*xs**2*x*eps**8                      *cos( - 2*gs - 4*ls + g + 3*l)
+ 5/8*del*xi**-2*kap**-1*sq2*xs*eta**4*eps**8 *cos(   gs - 2*ls + 3*l)
+ 15/8*del**2*xi**-2*kap**-1*eta**4*eps**8         *cos( - 3*ls + 3*l)
- 25/8*del*xi**-2*kap**-1*sq2*xs*eta**4*eps**8*cos( - gs - 4*ls + 3*l)
+ 3/8*del*sq2*xs*x**2*eps**8                       *cos(   gs + ls + 2*g + 2*l)
- 1/4*del**2*x**2*eps**8                           *cos(   2*g + 2*l)
+ 3/8*del*sq2*xs*x**2*eps**8                       *cos( - gs - ls + 2*g + 2*l)
+ 3/16*del*xi**-2*kap**-1*sq2*eta**4*x*eps**8 *cos( - ls + g + 2*l)
- 9/8*del*sq2*xs*y**2*eps**8                       *cos(   gs + ls + 2*h + 2*l)
+ 3/4*del**2*y**2*eps**8                           *cos(   2*h + 2*l)
- 9/8*del*sq2*xs*y**2*eps**8                       *cos( - gs - ls + 2*h + 2*l)
- 1/32*del*sq2*xs**3*eps**8                        *cos(   3*gs + ls + 2*l)
- 3/16*del*sq2*xs**3*eps**8                        *cos(   gs - ls + 2*l)
```

```
- 3/8*del*sq2*xs*y**2*eps**8                  *cos(   gs - ls + 2*l)
- 15/8*del*sq2*xs*x**2*eps**8                 *cos(   gs - ls + 2*l)
- 15/4*del**2*xs**2*eps**8                    *cos( - 2*ls + 2*l)
- 3/4*del**2*y**2*eps**8                      *cos( - 2*ls + 2*l)
- 15/4*del**2*x**2*eps**8                     *cos( - 2*ls + 2*l)
+ 195/16*del*sq2*xs**3*eps**8                 *cos( - gs - 3*ls + 2*l)
+ 21/8*del*sq2*xs*y**2*eps**8                 *cos( - gs - 3*ls + 2*l)
+ 105/8*del*sq2*xs*x**2*eps**8                *cos( - gs - 3*ls + 2*l)
+ 51/4*del**2*xs**2*eps**8                    *cos( - 2*gs - 4*ls + 2*l)
- 845/32*del*sq2*xs**3*eps**8                 *cos( - 3*gs - 5*ls + 2*l)
+ 45/16*del*xi**-2*kap**-1*sq2*eta**4*x*eps**8*cos( - 3*ls - g + 2*l)
+ 7/16*del*sq2*x**3*eps**8                    *cos(   2*ls + 3*g + l)
+ 3/4*del*sq2*y**2*x*eps**8                   *cos(   2*ls + 2*h + g + l)
+ 9/4*del*sq2*xs**2*x*eps**8                  *cos(   2*gs + 2*ls + g + l)
- 3/2*del**2*xs*x*eps**8                      *cos(   gs + ls + g + l)
+ 3/2*del*sq2*xs**2*x*eps**8                  *cos(   g + l)
- 3/2*del*sq2*y**2*x*eps**8                   *cos(   g + l)
- 1/4*del*sq2*x**3*eps**8                     *cos(   g + l)
- 3/2*del**2*xs*x*eps**8                      *cos( - gs - ls + g + l)
+ 9/4*del*sq2*xs**2*x*eps**8                  *cos( - 2*gs - 2*ls + g + l)
+ 3/4*del*sq2*y**2*x*eps**8                   *cos( - 2*ls - 2*h + g + l)
- 3/8*del*xi**-2*kap**-1*sq2*xs*eta**4*eps**8 *cos(   gs + l)
+ 9/8*del**2*xi**-2*kap**-1*eta**4*eps**8     *cos( - ls + l)
- 9/8*del*xi**-2*kap**-1*sq2*xs*eta**4*eps**8 *cos( - gs - 2*ls + l)
+ 9/4*del*sq2*y**2*x*eps**8                   *cos(   2*h - g + l)
+ 9/4*del**2*xs*x*eps**8                      *cos(   gs - ls - g + l)
- 45/4*del*sq2*xs**2*x*eps**8                 *cos( - 2*ls - g + l)
- 9/4*del*sq2*y**2*x*eps**8                   *cos( - 2*ls - g + l)
- 3*del*sq2*x**3*eps**8                       *cos( - 2*ls - g + l)
- 63/4*del**2*xs*x*eps**8                     *cos( - gs - 3*ls - g + l)
+ 153/4*del*sq2*xs**2*x*eps**8                *cos( - 2*gs - 4*ls - g + l)
- 105/8*del*sq2*xs*x**2*eps**8                *cos(   gs + 3*ls + 2*g)
+ 15/4*del**2*x**2*eps**8                     *cos(   2*ls + 2*g)
+ 15/8*del*sq2*xs*x**2*eps**8                 *cos( - gs + ls + 2*g)
+ 15/16*del*xi**-2*kap**-1*sq2*eta**4*x*eps**8*cos(   ls + g)
- 21/8*del*sq2*xs*y**2*eps**8                 *cos(   gs + 3*ls + 2*h)
+ 3/4*del**2*y**2*eps**8                      *cos(   2*ls + 2*h)
+ 3/8*del*sq2*xs*y**2*eps**8                  *cos( - gs + ls + 2*h)
- 53/16*del*sq2*xs**3*eps**8                  *cos(   3*gs + 3*ls)
+ 9/4*del**2*xs**2*eps**8                     *cos(   2*gs + 2*ls)
- 3/2*del*sq2*xs**3*eps**8                    *cos(   gs + ls)
+ 9/4*del*sq2*xs*y**2*eps**8                  *cos(   gs + ls)
- 9/4*del*sq2*xs*x**2*eps**8                  *cos(   gs + ls)
+ 3/4*del**2*xs**2*eps**8
- 3/4*del**2*y**2*eps**8
+ 3/4*del**2*x**2*eps**8
```

D *TRIP* code for the Lie transformation

Here is shown the code source that enabled to perform the Lie transformation and then go back into the unprimed variables.

D.1 Lie transformation and change of variables

The following *TRIP* code performs the 2 steps of the Lie transformation and then computes series that give the expressions of the double primed variables as a function of the unprimed ones, and reciprocally.

```
macro main[N]{
  _echo off$
  _modenum=NUMRATMP$
  _affdist=4$
  _cleanflag=0$
  _mode = POLP$
  1*eps$ // To sort series by order

  delpow=1$  //The order of the different small quantities
  xipow=1$
  etapow=2$
  xpow=1$
  ypow=1$
  xspow=1$
  kappow=4$
  yspow=5$

  %init_angle$
  nmax=15$
  count=0$
  dim les_phi[0:nmax,0:nmax]$ //holds the calculated phi_(j,p)
  dim est_calcules[0:nmax,0:nmax]$ //boolean matrix
  dim H[0:nmax]$ //holds the H_n
  dim W[0:nmax]$ //holds the W_n
  dim WT[0:1]$ WT[0]=0$ WT[1]=0$
  dim newH[0:nmax]$ //holds the transformed hamiltonians

  %derivation$
  efftronc$
  loadF(H1)$
  tel=({eps,N})$

  usetronc(tel)$
  H1=1*H1$
  efftronc$

  H1=mus**2*bets**3*Ls**-2*H1$

  H0=-mus**2*kap**2*xi**3*bets**3*eps**(2*kappow+3*xipow-4*etapow)
  /(2*eta**4*Ls**2)-mus**2*bets**3/(2*Ls**2)$
```

```
%initialisation$
%deprit$
usetronc(tel)$
H[1]=1*H[1]$
H[0]=1*H[0]$
R=H[0]+H[1]$
NH=%trans_lie["lk"]$

epsmin=puismin(NH-coef_ext(NH,(x,0),(y,0),(xs,0),(ys,0)),eps)$
H0=coef_ext(NH-coef_ext(NH,(x,0),(y,0),(xs,0),(ys,0)),(eps,epsmin))
*eps**(epsmin)+H[0];

H1=NH-H0$

%initialisation$
R=H[0]+H[1]$
NH2=%trans_lie["ghgs"]$

NH2=%simplifysq2[NH2]$
WT[0]=%simplifysq2[WT[0]]$
WT[1]=%simplifysq2[WT[1]]$

_path="resultats/o"+str(N)+"/"$
saveF(NH2)$

/*computation of the frequencies*/

nuL=%simplifysq2[%dL[NH2]]$stat(nuL)$
saveF(nuL)$
nuG=%simplifysq2[%dG[NH2]]$stat(nuG)$
saveF(nuG)$
nuH=%simplifysq2[%dH[NH2]]$stat(nuH)$
saveF(nuH)$
nuLs=%simplifysq2[%dLs[NH2]]$stat(nuLs)$
saveF(nuLs)$
nuGs=%simplifysq2[%dGs[NH2]]$stat(nuGs)$
saveF(nuGs)$
nuHs=%simplifysq2[%dHs[NH2]]$stat(nuHs)$
saveF(nuHs)$

/* double primed variables as a function
   of the unprimed ones */

lnew=%simplifysq2[%old_to_new[l]]$stat(lnew)$
saveF(lnew)$
gnew=%simplifysq2[%old_to_new[g]]$stat(gnew)$
saveF(gnew)$
hnew=%simplifysq2[%old_to_new[h]]$ stat(hnew)$
saveF(hnew)$
lsnew=%simplifysq2[%old_to_new[ls]]$stat(lsnew)$
saveF(lsnew)$
```

```
  gsnew=%simplifysq2[%old_to_new[gs]]$stat(gsnew)$
  saveF(gsnew)$
  hsnew=%simplifysq2[%old_to_new[hs]]$ stat(hsnew)$
  saveF(hsnew)$

  xnew=%simplifysq2[%old_to_new[x*eps**(xpow)]]$stat(xnew)$
  saveF(xnew)$
  ynew=%simplifysq2[%old_to_new[y*eps**(ypow)]]$stat(ynew)$
  saveF(ynew)$
  xsnew=%simplifysq2[%old_to_new[xs*eps**(xspow)]]$ stat(xsnew)$
  saveF(xsnew)$
  ysnew=%simplifysq2[%old_to_new[ys*eps**(yspow)]]$stat(ysnew)$
  saveF(ysnew)$
  etanew=%simplifysq2[%old_to_new[eta*eps**(etapow)]]$stat(etanew)$
  saveF(etanew)$
  Lsnew=%simplifysq2[%old_to_new[Ls]]$ stat(Lsnew)$
  saveF(Lsnew)$

  /* Unprimed variables as a function
     of the double primed ones*/

  lold=%simplifysq2[%new_to_old[l]]$stat(lold)$
  saveF(lold)$
  gold=%simplifysq2[%new_to_old[g]]$stat(gold)$
  saveF(gold)$
  hold=%simplifysq2[%new_to_old[h]]$ stat(hold)$
  saveF(hold)$
  lsold=%simplifysq2[%new_to_old[ls]]$stat(lsold)$
  saveF(lsold)$
  gsold=%simplifysq2[%new_to_old[gs]]$stat(gsold)$
  saveF(gsold)$
  hsold=%simplifysq2[%new_to_old[hs]]$ stat(hsold)$
  saveF(hsold)$

  xold=%simplifysq2[%new_to_old[x*eps**(xpow)]]$stat(xold)$
  saveF(xold)$
  yold=%simplifysq2[%new_to_old[y*eps**(ypow)]]$stat(yold)$
  saveF(yold)$
  xsold=%simplifysq2[%new_to_old[xs*eps**(xspow)]]$ stat(xsold)$
  saveF(xsold)$
  ysold=%simplifysq2[%new_to_old[ys*eps**(yspow)]]$stat(ysold)$
  saveF(ysold)$
  etaold=%simplifysq2[%new_to_old[eta*eps**(etapow)]]$stat(etaold)$
  saveF(etaold)$
  Lsold=%simplifysq2[%new_to_old[Ls]]$ stat(Lsold)$
  saveF(Lsold)$

}$

macro derivation{ /*This macro enables to define derivation wrt action
variables and thus the poisson bracket operation.*/
```

```
dxdL=-x*eps**(xpow-2*etapow)/(2*eta**2*Ls)$
dydL=-y*eps**(ypow-2*etapow)/(2*eta**2*Ls)$
detadL=1/(2*eta*eps**(etapow)*Ls)$

dxdG=1/(2*x*eta**2*eps**(xpow+2*etapow)*Ls)$

dydZ=1/(2*y*eta**2*eps**(ypow+2*etapow)*Ls)$

dxsdLs=-xs*eps**(xspow)/(2*Ls)$
dysdLs=-ys*eps**(yspow)/(2*Ls)$
detadLs=-eta*eps**(etapow)/(2*Ls)$

dxsdGs=1/(2*xs*eps**(xspow)*Ls)$

dysdZs=1/(2*ys*eps**(yspow)*Ls)$

macro dL[f]{
  return eps**(-xpow)*dxdL*deriv(f,x)+eps**(-ypow)*dydL*deriv(f,y)
  +eps**(-etapow)*detadL*deriv(f,eta)$
}$

macro dG[f]{
  return eps**(-xpow)*dxdG*deriv(f,x)$
}$

macro dH[f]{
  return eps**(-ypow)*dydZ*deriv(f,y)$
}$

macro dLs[f]{
  return eps**(-xspow)*dxsdLs*deriv(f,xs)+eps**(-yspow)*dysdLs*
  deriv(f,ys)+eps**(-etapow)*detadLs*deriv(f,eta)+deriv(f,Ls)$
}$

macro dGs[f]{
  return eps**(-xspow)*dxsdGs*deriv(f,xs)$
}$

macro dHs[f]{
  return eps**(-yspow)*dysdZs*deriv(f,ys)$
}$

macro crochet[f1,f2]{ //The poisson bracket
  private res$
  res=%dL[f1]*deriv(f2,l)-deriv(f1,l)*%dL[f2]$
  res=res+%dG[f1]*deriv(f2,g)-deriv(f1,g)*%dG[f2]$
  res=res+%dH[f1]*deriv(f2,h)-deriv(f1,h)*%dH[f2]$
  res=res+%dLs[f1]*deriv(f2,ls)-deriv(f1,ls)*%dLs[f2]$
  res=res+%dGs[f1]*deriv(f2,gs)-deriv(f1,gs)*%dGs[f2]$
  res=res+%dHs[f1]*deriv(f2,hs)-deriv(f1,hs)*%dHs[f2]$
  return res$
}$
```

```
}$

macro init_angle{
  ll=expi(l,1,0)$
  gg=expi(g,1,0)$
  hh=expi(h,1,0)$
  lls=expi(ls,1,0)$
  ggs=expi(gs,1,0)$
  hhs=expi(hs,1,0)$

  lls0=expi(ls0,1,0)$
  hhs0=expi(hs0,1,0)$
  ggs0=expi(gs0,1,0)$
  nnuK=expi(nuKt,1,0)$
  nnuH=expi(nuHt,1,0)$
  nnuG=expi(nuGt,1,0)$
  nnuL=expi(nuLt,1,0)$
  ll0=expi(l0,1,0)$
  gg0=expi(g0,1,0)$
  hh0=expi(h0,1,0)$
  lln0=expi(ln0,1,0)$
  ggn0=expi(gn0,1,0)$
  hhn0=expi(hn0,1,0)$
}$

macro initialisation{

  private a$
  private b$

  H[0]=H0$
  H[1]=H1$

  for a=2 to nmax{H[a]=0$}$

  for a=0 to nmax{
    for b=0 to nmax{
      est_calcules[a,b]=0$
      les_phi[a,b]=0$
    }$
  }$

  for a=0 to nmax{
    est_calcules[a,0]=1$
    les_phi[a,0]=H[a]$
    W[a]=0$
    newH[a]=0$
  }$

}$
```

```
macro deprit{ //compute the phi_j^p of the Deprit's formula
  macro phi_aux[j]{ // auxiliary macro to deprit
    private Z$
    if (j==0) then {
      return H[0]$
      stop$
    }
    else {
      if(est_calcules[j,j]==1) then {
        return les_phi[j,j]$
        stop$
      }
      else {
        Z=%crochet[W[1],%phi_aux[j-1]]$
        est_calcules[j,j]=1$
        les_phi[j,j]=Z$
        return Z$
        stop$
      }$
    }$
  }$

  macro phi[j,p]{ // calcule les phi_j^p
    private Z$
    private a$
    private b$
    b=j-p+1$
    if (j==p) then {
      return %phi_aux[j]$
      stop$
    }
    else {
      if(est_calcules[j,p]==1) then {
        return les_phi[j,p]$
        stop$
      }
      else {
        Z=0$
        for a=1 to b {
          Z=Z+%crochet[W[a],%phi[j-a,p-1]]$
        }$
        est_calcules[j,p]=1$
        les_phi[j,p]=Z$
        return Z$
        stop$
      }$
    }$
  }$
}$

macro psi[c]{ //Compute the psi_c of deprit's formula
```

```
  private res$
  private a$
  res=H[c]$
  for a=1 to c-1 {
    res=res+%crochet[W[a],H[c-a]]$
  }$
  for a=2 to c {
    res=res+%phi[c,a]/fac(a)$
  }$
  return res$
}$

macro trans_lie[drapeau]{
  private NH$
  private NHa$
  private pour_resolution$
  private res, a_rajouter$
  private a$
  private k1,k2,k3,k4$
  private dmin1, dmin2, dmax1, dmax2, dmin3, dmin4, dmax3, dmax4$
  private var,var1,var2,var3,var4,nu1,nu2,nu3,nu4,nu$
  switch(drapeau){
    case "lk" : { // k=ls
      var1=ll$
      var2=lls$
      nu1=%dL[H[0]];
      nu2=%dLs[H[0]];
      for a=1 to nmax{
        a;
        pour_resolution=%psi[a]$
        pour_resolution=%simplifysq2[pour_resolution]$
        NHa=coef_ext(pour_resolution,(var1,0),(var2,0))$
        newH[a]=NHa$
        pour_resolution=pour_resolution-NHa$
        res=0$
        dmin1=puismin(pour_resolution,var1)$
        dmax1=puismax(pour_resolution,var1)$
        dmin2=puismin(pour_resolution,var2)$
        dmax2=puismax(pour_resolution,var2)$
        for k1=dmin1 to dmax1{
          for k2=dmin2 to dmax2{
            if(k1!=0 || k2!=0) then {
              res=res+coef_ext(pour_resolution,(var1,k1),(var2,k2))
              *var1^k1*var2^k2/(i*(k1*nu1+k2*nu2))$
            }$
          }$
        }$
        res=1*res$
        W[a]=res$
        WT[count]=WT[count]+1*W[a]$
        nmaxlk=nmax$
        if(res==0) then {nmaxlk=a$ stop;}$
```

```
  }$
}$
case "ghgs" : {
  var1=gg$
  var2=hh$
  var3=ggs$
  nu1=%dG[H[0]];
  nu2=%dH[H[0]];
  nu3=%dGs[H[0]];
  for a=1 to nmax{
    a;
    pour_resolution=%psi[a]$
    pour_resolution=%simplifysq2[pour_resolution]$
    res=0$
    dmin1=puismin(pour_resolution,var1)$
    dmax1=puismax(pour_resolution,var1)$
    dmin2=puismin(pour_resolution,var2)$
    dmax2=puismax(pour_resolution,var2)$
    dmin3=puismin(pour_resolution,var3)$
    dmax3=puismax(pour_resolution,var3)$
    NHa=coef_ext(pour_resolution,(var1,0),(var2,0),(var3,0))$
    for k1=dmin1 to dmax1 {
      for k2=dmin2 to dmax2 {
        for k3=dmin3 to dmax3 {
          if(k1*nu1+k2*nu2+k3*nu3==0 && (k1!=0 || k2!=0 || k3!=0 )) then {
            a_rajouter=coef_ext(pour_resolution,(var1,k1),(var2,k2),(var3,k3));
            NHa=NHa+a_rajouter*var1^k1*var2^k2*var3^k3$
            if(a_rajouter!=0) then {
              NHa;
            };
          }$
        }$
      }$
    }$
    pour_resolution=pour_resolution-NHa$
    for k1=dmin1 to dmax1 {
      for k2=dmin2 to dmax2 {
        for k3=dmin3 to dmax3 {
          if(k1*nu1+k2*nu2+k3*nu3+k4*nu4!=0 && (k1!=0 || k2!=0 || k3!=0)) then {
            a_rajouter=coef_ext(pour_resolution,(var1,k1),(var2,k2),(var3,k3))$
            res=res+a_rajouter*var1^k1*var2^k2*var3^k3/(i*(k1*nu1+k2*nu2+k3*nu3))$
          }$
        }$
      }$
    }$
    newH[a]=NHa$
    res=1*res$
    W[a]=res$
    WT[count]=WT[count]+1*W[a]$
    nmaxgh=nmax$
    if(res==0) then {nmaxgh=a$ stop;}$
  }$
```

```
    }$
    case "hs" :{
      var=hhs$
      nu=%dHs[H[0]]$
      for a=1 to nmax{
        a;
        pour_resolution=%psi[a]$
        pour_resolution=%simplifysq2[pour_resolution]$
        NHa=coef_ext(pour_resolution,(var1,0),(var2,0))$
        newH[a]=NHa$
        pour_resolution=pour_resolution-NHa$
        res=0$
        dmin1=puismin(pour_resolution,var)$
        dmax1=puismax(pour_resolution,var)$
        for k1=dmin1 to dmax1 {
          if (k1!=0) then {
            res=res+coef_ext(pour_resolution,(var,k1))*var^k1/(i*(k1*nu1))$
          }$
        }$
        res=1*res$
        W[a]=res$
        WT[count]=WT[count]+1*W[a]$
        nmaxhs=nmax$
        if(res==0) then {nmaxhs=a$ stop;}$
      }$
      nmax=max(nmaxlk,nmaxgh,nmaxhs)$
    }$
  }$
  NH=H[0]$
  for a=1 to nmax{
    newH[a]$
    NH=NH+newH[a]$
  }$
  count=count+1$
  return %simplifysq2[NH]$
}$

macro trans_all[gen,old]{
  private q, new, newT$
  dim new[0:nmax]$
  new[0]=old$
  newT=0$
  for q=1 to nmax{
    new[q]=%crochet[gen,new[q-1]]$
  }$
  for q=0 to nmax{
    newT=newT+1*new[q]/fac(q)$
  }$
  return newT$
}$

macro old_to_new[var]{
```

```
  private aux$
  aux=var$
  aux=%trans_all[-WT[1],aux]$
  aux=%trans_all[-WT[0],aux]$
  return aux$
}$

macro new_to_old[var]{
  private aux$
  aux=var$
  aux=%trans_all[WT[0],aux]$
  aux=%trans_all[WT[1],aux]$
  return aux$
}$
```

D.2 Integration of $\mathcal{H}''$

The following *TRIP* code integrates the trivially integrable hamiltonian $\mathcal{H}''$

```
macro main[N]{

  _echo off$
  _modenum=NUMRATMP$
  _affdist=4$
  _cleanflag=0$
  _mode = POLP$

  _path="resultats/o"+str(N)+"/"$

  %init_angle$

  loadF(nuL)$
  loadF(nuG)$
  loadF(nuH)$
  loadF(nuLs)$
  loadF(nuGs)$
  loadF(nuHs)$

  loadF(lnew)$
  loadF(gnew)$
  loadF(hnew)$
  loadF(lsnew)$
  loadF(gsnew)$
  loadF(hsnew)$

  loadF(xnew)$
  loadF(ynew)$
  loadF(xsnew)$
  loadF(ysnew)$
  loadF(etanew)$
  loadF(Lsnew)$
```

```
  loadF(lold)$
  loadF(gold)$
  loadF(hold)$
  loadF(lsold)$
  loadF(gsold)$
  loadF(hsold)$

  loadF(xold)$
  loadF(yold)$
  loadF(xsold)$
  loadF(ysold)$
  loadF(etaold)$
  loadF(Lsold)$

  msg("Series are loaded in ram")$

  %integration$

  saveF(lold0)$
  saveF(gold0)$
  saveF(hold0)$
  saveF(lsold0)$
  saveF(gsold0)$
  saveF(hsold0)$

  saveF(xold0)$
  saveF(yold0)$
  saveF(xsold0)$
  saveF(ysold0)$
  saveF(etaold0)$
  saveF(Lsold0)$


}$

macro init_angle{

  ll=expi(l,1,0)$
  gg=expi(g,1,0)$
  hh=expi(h,1,0)$
  lls=expi(ls,1,0)$
  ggs=expi(gs,1,0)$
  hhs=expi(hs,1,0)$

  lln0=expi(ln0,1,0)$
  ggn0=expi(gn0,1,0)$
  hhn0=expi(hn0,1,0)$
  llsn0=expi(lsn0,1,0)$
  ggsn0=expi(gsn0,1,0)$
  hhsn0=expi(hsn0,1,0)$
  nnuL=expi(nuLt,1,0)$
  nnuG=expi(nuGt,1,0)$
```

```
  nnuH=expi(nuHt,1,0)$
  nnuLs=expi(nuLst,1,0)$
  nnuGs=expi(nuGst,1,0)$
  nnuHs=expi(nuHst,1,0)$

}$

macro integration{

  efftronc$

  varll=lln0*nnuL$  //nnuL=exp(i*t*nuL)
  vargg=ggn0*nnuG$
  varhh=hhn0*nnuH$
  varlls=llsn0*nnuLs$
  varggs=ggsn0*nnuGs$
  varhhs=hhsn0*nnuHs$

  varx=xn0$
  vary=yn0$
  varxs=xsn0$
  varys=ysn0$
  vareta=etan0$
  varLs=Lsn0$

  varl=ln0+nuL*t$
  varg=gn0+nuG*t$
  varh=hn0+nuH*t$
  varls=lsn0+nuLs*t$
  vargs=gsn0+nuGs*t$
  varhs=hsn0+nuHs*t$

  dimvar  tabv[0:17]$
  dim  tabexpr[0:17]$

  tabv[0]:=ll$  tabexpr[0]=varll$
  tabv[1]:=gg$  tabexpr[1]=vargg$
  tabv[2]:=hh$  tabexpr[2]=varhh$
  tabv[3]:=lls$  tabexpr[3]=varlls$
  tabv[4]:=ggs$  tabexpr[4]=varggs$
  tabv[5]:=hhs$  tabexpr[5]=varhhs$
  tabv[6]:=l$  tabexpr[6]=varl$
  tabv[7]:=g$  tabexpr[7]=varg$
  tabv[8]:=h$  tabexpr[8]=varh$
  tabv[9]:=ls$  tabexpr[9]=varls$
  tabv[10]:=gs$  tabexpr[10]=vargs$
  tabv[11]:=hs$  tabexpr[11]=varhs$
  tabv[12]:=x$  tabexpr[12]=varx$
  tabv[13]:=y$  tabexpr[13]=vary$
  tabv[14]:=xs$  tabexpr[14]=varxs$
  tabv[15]:=ys$  tabexpr[15]=varys$
  tabv[16]:=eta$  tabexpr[16]=vareta$
```

```
  tabv[17]:=Ls$  tabexpr[17]=varLs$

  lold0=subst(lold,tabv,tabexpr)$
  stat(lold0)$
  gold0=subst(gold,tabv,tabexpr)$
  stat(gold0)$
  hold0=subst(hold,tabv,tabexpr)$
  stat(hold0)$
  lsold0=subst(lsold,tabv,tabexpr)$
  stat(lsold0)$
  gsold0=subst(gsold,tabv,tabexpr)$
  stat(gsold0)$
  hsold0=subst(hsold,tabv,tabexpr)$
  stat(hsold0)$

  xold0=subst(xold,tabv,tabexpr)$
  stat(xold0)$
  yold0=subst(yold,tabv,tabexpr)$
  stat(yold0)$
  xsold0=subst(xsold,tabv,tabexpr)$
  stat(xsold0)$
  ysold0=subst(ysold,tabv,tabexpr)$
  stat(ysold0)$
  etaold0=subst(etaold,tabv,tabexpr)$
  stat(etaold0)$
  Lsold0=subst(Lsold,tabv,tabexpr)$
  stat(Lsold0)$


}$
```

E Explicit expression of W, W' and $\mathcal{H}''$

Here, we display the expressions of $\mathcal{H}''$, W and W' up to the 10^{th}, the 8^{th} and the 10^{th} order, respectively.

As in appendix C, ε (eps) is a fictional variable whose power indicates the order of the term while *sq2* and *del* are $\sqrt{2}$ and δ respectively.

$$\frac{L_s^2}{\mu_s^2\beta_s^3}\mathcal{H}'' =$$

```
- 1/2
- 1/2*xi**3*kap**2*eta**-4*eps**3
- 1/4*del*xi**-4*kap**-2*eta**8*eps**5
+ 1/4*del**2*xi**-4*kap**-2*eta**8*eps**6
- 3/4*del*xi**-4*kap**-2*xs**2*eta**8*eps**7
+ 3/4*del*xi**-4*kap**-2*y**2*eta**8*eps**7
+ 49/64*del**2*xi**-11*kap**-6*eta**20*eps**7
- 3/4*del*xi**-4*kap**-2*eta**8*x**2*eps**7
```

```
+ 3/4*del**2*xi**-4*kap**-2*xs**2*eta**8*eps**8
- 3/4*del**2*xi**-4*kap**-2*y**2*eta**8*eps**8
- 9/32*del**2*xi**-8*kap**-4*y**2*eta**14*eps**8
- 49/32*del**3*xi**-11*kap**-6*eta**20*eps**8
+ 97/32*del**2*xi**-14*kap**-8*eta**26*eps**8
+ 3/4*del**2*xi**-4*kap**-2*eta**8*x**2*eps**8
- 225/32*del**2*xi**-8*kap**-4*eta**14*x**2*eps**8
- 3/2*del*xi**-4*kap**-2*xs**4*eta**8*eps**9
+ 9/4*del*xi**-4*kap**-2*xs**2*y**2*eta**8*eps**9
- 3/8*del*xi**-4*kap**-2*y**4*eta**8*eps**9
+ 9/16*del**3*xi**-8*kap**-4*y**2*eta**14*eps**9
- 9/64*del*xi**-8*kap**-4*eta**16*eps**9
+ 49/64*del**4*xi**-11*kap**-6*eta**20*eps**9
+ 735/64*del**2*xi**-11*kap**-6*xs**2*eta**20*eps**9
+ 27/128*del**3*xi**-12*kap**-6*y**2*eta**20*eps**9
- 51/32*del**2*xi**-11*kap**-6*y**2*eta**20*eps**9
- 97/16*del**3*xi**-14*kap**-8*eta**26*eps**9
+ 135/64*del**3*xi**-18*kap**-10*eta**32*eps**9
+ 1477/192*del**2*xi**-17*kap**-10*eta**32*eps**9
- 9/4*del*xi**-4*kap**-2*xs**2*eta**8*x**2*eps**9
+ 3*del*xi**-4*kap**-2*y**2*eta**8*x**2*eps**9
+ 225/16*del**3*xi**-8*kap**-4*eta**14*x**2*eps**9
- 675/128*del**3*xi**-12*kap**-6*eta**20*x**2*eps**9
- 873/32*del**2*xi**-11*kap**-6*eta**20*x**2*eps**9
+ 3/8*del*xi**-4*kap**-2*eta**8*x**4*eps**9
+ 3/2*del**2*xi**-4*kap**-2*xs**4*eta**8*eps**10
- 9/4*del**2*xi**-4*kap**-2*xs**2*y**2*eta**8*eps**10
+ 3/8*del**2*xi**-4*kap**-2*y**4*eta**8*eps**10
- 9/32*del**4*xi**-8*kap**-4*y**2*eta**14*eps**10
- 33/16*del**2*xi**-8*kap**-4*xs**2*y**2*eta**14*eps**10
+ 27/64*del**2*xi**-8*kap**-4*y**4*eta**14*eps**10
+ 9/32*del**2*xi**-8*kap**-4*eta**16*eps**10
- 735/32*del**3*xi**-11*kap**-6*xs**2*eta**20*eps**10
- 81/128*del**4*xi**-12*kap**-6*y**2*eta**20*eps**10
+ 51/16*del**3*xi**-11*kap**-6*y**2*eta**20*eps**10
- 225/512*del**2*xi**-12*kap**-6*eta**22*eps**10
+ 97/32*del**4*xi**-14*kap**-8*eta**26*eps**10
+ 2619/32*del**2*xi**-14*kap**-8*xs**2*eta**26*eps**10
- 405/2048*del**4*xi**-16*kap**-8*y**2*eta**26*eps**10
+ 153/128*del**3*xi**-15*kap**-8*y**2*eta**26*eps**10
- 203/32*del**2*xi**-14*kap**-8*y**2*eta**26*eps**10
- 405/64*del**4*xi**-18*kap**-10*eta**32*eps**10
- 1477/96*del**3*xi**-17*kap**-10*eta**32*eps**10
+ 39/4*del**3*xi**-21*kap**-12*eta**38*eps**10
+ 2507/144*del**2*xi**-20*kap**-12*eta**38*eps**10
+ 9/4*del**2*xi**-4*kap**-2*xs**2*eta**8*x**2*eps**10
- 3*del**2*xi**-4*kap**-2*y**2*eta**8*x**2*eps**10
- 225/32*del**4*xi**-8*kap**-4*eta**14*x**2*eps**10
- 825/16*del**2*xi**-8*kap**-4*xs**2*eta**14*x**2*eps**10
+ 189/16*del**2*xi**-8*kap**-4*y**2*eta**14*x**2*eps**10
+ 2025/128*del**4*xi**-12*kap**-6*eta**20*x**2*eps**10
+ 873/16*del**3*xi**-11*kap**-6*eta**20*x**2*eps**10
```

```
- 58725/2048*del**4*xi**-16*kap**-8*eta**26*x**2*eps**10
- 4995/128*del**3*xi**-15*kap**-8*eta**26*x**2*eps**10
- 76*del**2*xi**-14*kap**-8*eta**26*x**2*eps**10
- 3/8*del**2*xi**-4*kap**-2*eta**8*x**4*eps**10
+ 675/64*del**2*xi**-8*kap**-4*eta**14*x**4*eps**10
```

$$W =$$

```
- 3/8*del*xi**-7*kap**-4*Ls*eta**14*eps**6                  *sin( - 2*ls + 2*l)
- 3/4*del*xi**-4*kap**-2*sq2*xs*Ls*eta**8*eps**6            *sin(   gs + ls)
- 1/4*del*xi**-7*kap**-4*sq2*Ls*eta**14*x*eps**7            *sin( - 2*ls + g + 3*l)
+ 3/16*del*xi**-7*kap**-4*sq2*xs*Ls*eta**14*eps**7          *sin(   gs - ls + 2*l)
+ 3/8*del**2*xi**-7*kap**-4*Ls*eta**14*eps**7               *sin( - 2*ls + 2*l)
- 3/8*del*xi**-10*kap**-6*Ls*eta**20*eps**7                 *sin( - 2*ls + 2*l)
- 21/16*del*xi**-7*kap**-4*sq2*xs*Ls*eta**14*eps**7         *sin( - gs - 3*ls + 2*l)
+ 1/2*del*xi**-7*kap**-4*sq2*Ls*eta**14*x*eps**7            *sin(   g + l)
+ 9/4*del*xi**-7*kap**-4*sq2*Ls*eta**14*x*eps**7            *sin( - 2*ls - g + l)
- 15/8*del*xi**-4*kap**-2*Ls*eta**8*x**2*eps**7             *sin(   2*ls + 2*g)
- 3/8*del*xi**-4*kap**-2*y**2*Ls*eta**8*eps**7              *sin(   2*ls + 2*h)
- 9/8*del*xi**-4*kap**-2*xs**2*Ls*eta**8*eps**7             *sin(   2*gs + 2*ls)
+ 3/4*del**2*xi**-4*kap**-2*sq2*xs*Ls*eta**8*eps**7         *sin(   gs + ls)
- 3/8*del*xi**-7*kap**-4*Ls*eta**14*x**2*eps**8             *sin( - 2*ls + 2*g + 4*l)
- 9/256*del**2*xi**-14*kap**-8*Ls*eta**26*eps**8            *sin( - 4*ls + 4*l)
+ 1/4*del*xi**-7*kap**-4*xs*Ls*eta**14*x*eps**8             *sin(   gs - ls + g + 3*l)
+ 1/4*del**2*xi**-7*kap**-4*sq2*Ls*eta**14*x*eps**8         *sin( - 2*ls + g + 3*l)
- 1/6*del*xi**-10*kap**-6*sq2*Ls*eta**20*x*eps**8           *sin( - 2*ls + g + 3*l)
- 7/4*del*xi**-7*kap**-4*xs*Ls*eta**14*x*eps**8             *sin( - gs - 3*ls + g + 3*l)
- 5/24*del*xi**-9*kap**-5*Ls*eta**18*eps**8                 *sin( - 3*ls + 3*l)
- 15/64*del**2*xi**-11*kap**-6*sq2*Ls*eta**20*x*eps**8      *sin( - 4*ls - g + 3*l)
+ 1/8*del*xi**-7*kap**-4*Ls*eta**14*x**2*eps**8             *sin(   2*g + 2*l)
- 3/8*del*xi**-7*kap**-4*y**2*Ls*eta**14*eps**8             *sin(   2*h + 2*l)
- 3/16*del**2*xi**-7*kap**-4*sq2*xs*Ls*eta**14*eps**8       *sin(   gs - ls + 2*l)
+ 9/16*del**2*xi**-11*kap**-6*sq2*xs*Ls*eta**20*eps**8      *sin(   gs - ls + 2*l)
+ 3/32*del*xi**-10*kap**-6*sq2*xs*Ls*eta**20*eps**8         *sin(   gs - ls + 2*l)
+ 15/8*del*xi**-7*kap**-4*xs**2*Ls*eta**14*eps**8           *sin( - 2*ls + 2*l)
+ 3/8*del*xi**-7*kap**-4*y**2*Ls*eta**14*eps**8             *sin( - 2*ls + 2*l)
+ 3/8*del**2*xi**-10*kap**-6*Ls*eta**20*eps**8              *sin( - 2*ls + 2*l)
- 1/4*del**2*xi**-14*kap**-8*Ls*eta**26*eps**8              *sin( - 2*ls + 2*l)
- 3/8*del*xi**-13*kap**-8*Ls*eta**26*eps**8                 *sin( - 2*ls + 2*l)
+ 15/8*del*xi**-7*kap**-4*Ls*eta**14*x**2*eps**8            *sin( - 2*ls + 2*l)
+ 21/16*del**2*xi**-7*kap**-4*sq2*xs*Ls*eta**14*eps**8      *sin( - gs - 3*ls + 2*l)
- 9/16*del**2*xi**-11*kap**-6*sq2*xs*Ls*eta**20*eps**8      *sin( - gs - 3*ls + 2*l)
- 63/32*del*xi**-10*kap**-6*sq2*xs*Ls*eta**20*eps**8        *sin( - gs - 3*ls + 2*l)
- 51/8*del*xi**-7*kap**-4*xs**2*Ls*eta**14*eps**8           *sin( - 2*gs - 4*ls + 2*l)
+ 3/2*del*xi**-7*kap**-4*xs*Ls*eta**14*x*eps**8             *sin(   gs + ls + g + l)
- 1/2*del**2*xi**-7*kap**-4*sq2*Ls*eta**14*x*eps**8         *sin(   g + l)
+ 135/64*del**2*xi**-11*kap**-6*sq2*Ls*eta**20*x*eps**8     *sin(   g + l)
+ 3/2*del*xi**-7*kap**-4*xs*Ls*eta**14*x*eps**8             *sin( - gs - ls + g + l)
- 3/8*del*xi**-9*kap**-5*Ls*eta**18*eps**8                  *sin( - ls + l)
- 9/4*del*xi**-7*kap**-4*xs*Ls*eta**14*x*eps**8             *sin(   gs - ls - g + l)
- 9/4*del**2*xi**-7*kap**-4*sq2*Ls*eta**14*x*eps**8         *sin( - 2*ls - g + l)
+ 15/32*del**2*xi**-11*kap**-6*sq2*Ls*eta**20*x*eps**8      *sin( - 2*ls - g + l)
```

```
+ 9/2*del*xi**-10*kap**-6*sq2*Ls*eta**20*x*eps**8         *sin( - 2*ls - g + l)
+ 63/4*del*xi**-7*kap**-4*xs*Ls*eta**14*x*eps**8          *sin( - gs - 3*ls - g + l)
- 35/8*del*xi**-4*kap**-2*sq2*xs*Ls*eta**8*x**2*eps**8    *sin(   gs + 3*ls + 2*g)
+ 15/8*del**2*xi**-4*kap**-2*Ls*eta**8*x**2*eps**8        *sin(   2*ls + 2*g)
- 45/32*del**2*xi**-8*kap**-4*Ls*eta**14*x**2*eps**8      *sin(   2*ls + 2*g)
+ 15/8*del*xi**-4*kap**-2*sq2*xs*Ls*eta**8*x**2*eps**8    *sin( - gs + ls + 2*g)
+ 15/16*del*xi**-6*kap**-3*sq2*Ls*eta**12*x*eps**8        *sin(   ls + g)
- 7/8*del*xi**-4*kap**-2*sq2*xs*y**2*Ls*eta**8*eps**8     *sin(   gs + 3*ls + 2*h)
+ 3/8*del**2*xi**-4*kap**-2*y**2*Ls*eta**8*eps**8         *sin(   2*ls + 2*h)
+ 9/32*del**2*xi**-8*kap**-4*y**2*Ls*eta**14*eps**8       *sin(   2*ls + 2*h)
+ 3/8*del*xi**-4*kap**-2*sq2*xs*y**2*Ls*eta**8*eps**8     *sin( - gs + ls + 2*h)
- 53/48*del*xi**-4*kap**-2*sq2*xs**3*Ls*eta**8*eps**8     *sin(   3*gs + 3*ls)
+ 9/8*del**2*xi**-4*kap**-2*xs**2*Ls*eta**8*eps**8        *sin(   2*gs + 2*ls)
- 3/2*del*xi**-4*kap**-2*sq2*xs**3*Ls*eta**8*eps**8       *sin(   gs + ls)
+ 9/4*del*xi**-4*kap**-2*sq2*xs*y**2*Ls*eta**8*eps**8     *sin(   gs + ls)
+ 147/32*del**2*xi**-11*kap**-6*sq2*xs*Ls*eta**20*eps**8*sin(   gs + ls)
- 9/4*del*xi**-4*kap**-2*sq2*xs*Ls*eta**8*x**2*eps**8     *sin(   gs + ls)
```

$$W' =$$

```
+ 5/4*y**2*Ls*eta**2*x**2*eps**8                                      *sin( - 2*h + 2*g)
- 5/2*xi**-2*kap**-1*xs*Ls*eta**6*x*eps**8                            *sin( - gs + g)
+ 45/16*del*xi**-4*kap**-2*xs**2*Ls*eta**8*x**2*eps**9                *sin( - 2*gs + 2*g)
- 135/32*del*xi**-4*kap**-2*y**2*Ls*eta**8*x**2*eps**9                *sin( - 2*h + 2*g)
+ 5*del*xi**-2*kap**-1*xs*Ls*eta**6*x*eps**9                          *sin( - gs + g)
+ 45/4*del*xi**-6*kap**-3*xs*Ls*eta**12*x*eps**9                      *sin( - gs + g)
+ 9/16*del*xi**-4*kap**-2*xs**2*y**2*Ls*eta**8*eps**9                 *sin( - 2*gs + 2*h)
- 45/16*del**2*xi**-4*kap**-2*xs**2*Ls*eta**8*x**2*eps**10            *sin( - 2*gs + 2*g)
+ 3735/128*del**2*xi**-8*kap**-4*xs**2*Ls*eta**14*x**2*eps**10*sin( - 2*gs + 2*g)
+ 333/16*del*xi**-7*kap**-4*xs**2*Ls*eta**14*x**2*eps**10             *sin( - 2*gs + 2*g)
+ 5/2*y**4*Ls*eta**2*x**2*eps**10                                     *sin( - 2*h + 2*g)
+ 135/32*del**2*xi**-4*kap**-2*y**2*Ls*eta**8*x**2*eps**10            *sin( - 2*h + 2*g)
+ 1125/64*del**2*xi**-8*kap**-4*y**2*Ls*eta**14*x**2*eps**10          *sin( - 2*h + 2*g)
- 953/64*del*xi**-7*kap**-4*y**2*Ls*eta**14*x**2*eps**10              *sin( - 2*h + 2*g)
- 5/4*y**2*Ls*eta**2*x**4*eps**10                                     *sin( - 2*h + 2*g)
- 35/8*xi**-2*kap**-1*xs**3*Ls*eta**6*x*eps**10                       *sin( - gs + g)
+ 15/4*xi**-2*kap**-1*xs*y**2*Ls*eta**6*x*eps**10                     *sin( - gs + g)
- 135/4*del**2*xi**-6*kap**-3*xs*Ls*eta**12*x*eps**10                 *sin( - gs + g)
- 19935/128*del**2*xi**-10*kap**-5*xs*Ls*eta**18*x*eps**10            *sin( - gs + g)
+ 2945/32*del*xi**-9*kap**-5*xs*Ls*eta**18*x*eps**10                  *sin( - gs + g)
- 45/8*xi**-2*kap**-1*xs*Ls*eta**6*x**3*eps**10                       *sin( - gs + g)
+ 5/48*xi**-2*kap**-1*xs*y**2*Ls*eta**6*x*eps**10                     *sin(   gs - 2*h + g)
- 9/16*del**2*xi**-4*kap**-2*xs**2*y**2*Ls*eta**8*eps**10             *sin( - 2*gs + 2*h)
- 315/128*del**2*xi**-8*kap**-4*xs**2*y**2*Ls*eta**14*eps**10         *sin( - 2*gs + 2*h)
+ 51/16*del*xi**-7*kap**-4*xs**2*y**2*Ls*eta**14*eps**10              *sin( - 2*gs + 2*h)
- 2*ys*y*Ls*eta**2*eps**10                                            *sin( - hs + h)
```

References

Andoyer, H. (1923). Cours de Mecanique Celeste. en. *Paris*,

Boué, G. and Laskar, J. (2006). Precession of a Planet with a Satellite. en. *Icarus*, 185.2, p. 312.

Delaunay, C. (1860). *Théorie du mouvement de la lune*. fr. Didot.

Deprit, A. (1969). Canonical Transformations Depending on a Small Parameter. en. *Celestial Mechanics*, 1, p. 12.

Gastineau, M. and Laskar, J. (2011). TRIP: A Computer Algebra System Dedicated to Celestial Mechanics and Perturbation Series. en. *ACM SIGSAM Bulletin*, 44.3/4, p. 194.

Laskar, J. (2007). Cours de dynamique des systèmes planétaires (Unpublished). fr, p. 70.

Laskar, J. and Robutel, P. (1995). Stability of the Planetary Three-Body Problem. I. Expansion of the Planetary Hamiltonian. en. *Celestial Mechanics and Dynamical Astronomy*, 62.3, p. 193.

LI, Z. (2018). Lunar theory in the footsteps of Charles-Eugène Delaunay (Unpublished). en, p. 47.

Mignard, F. (1980). The Evolution of the Lunar Orbit Revisited - Part Two. en. *Moon and Planets*, 23.2, p. 185.

Neron de Surgy, O. and Laskar, J. (1997). On the Long Term Evolution of the Spin of the Earth. en. *Astronomy and Astrophysics*, 318, p. 975.