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Evection resonance in the 3-body problem

The harmonic exomoon

or how celestial mechanics is easy

Jérémy Couturier

Exoplanet seminar

February 21st, 2025

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Do exoplanets have exomoons ?

- Exomoons orbit their exoplanet
- They are perturbed by the star
- If they orbit too close $\rightarrow$ Torn apart by tides
- If they orbit too far $\rightarrow$ An instability triggers

This instability is due to a resonance called the evection resonance.

- The gap between the Roche limit and the instability can be small
- I will focus on this instability and how it affects exomoons
- Can be recovered from a simple harmonic oscillator !

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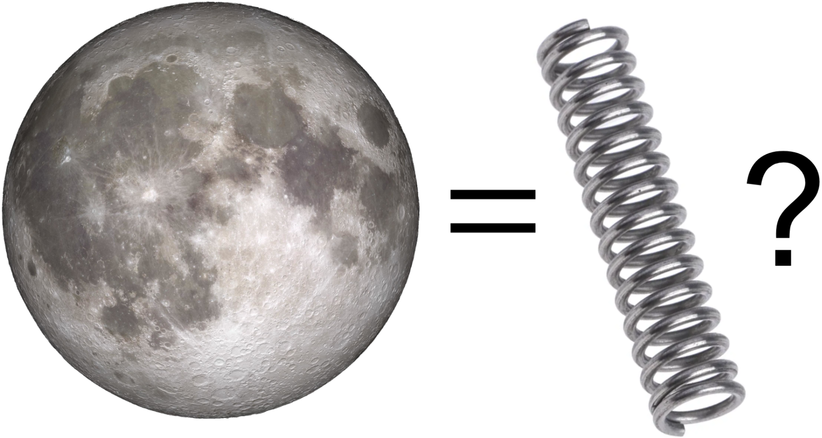
Are exomoons just harmonic oscillators ?



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Two instabilities exist

- The first one occurs close to the planet and is due to J_2
- It mostly affects satellite formation
- Very well know by people focused on the formation of the Moon
- Mostly irrelevant for exomoons
- I focused on it while studying the Moon formation

The other instability is much less known

- It is only due to perturbations by the Star
- It is very relevant for exomoons
- These are not my researches, these are previous results that I present in a completely novel way (with an harmonic oscillator).

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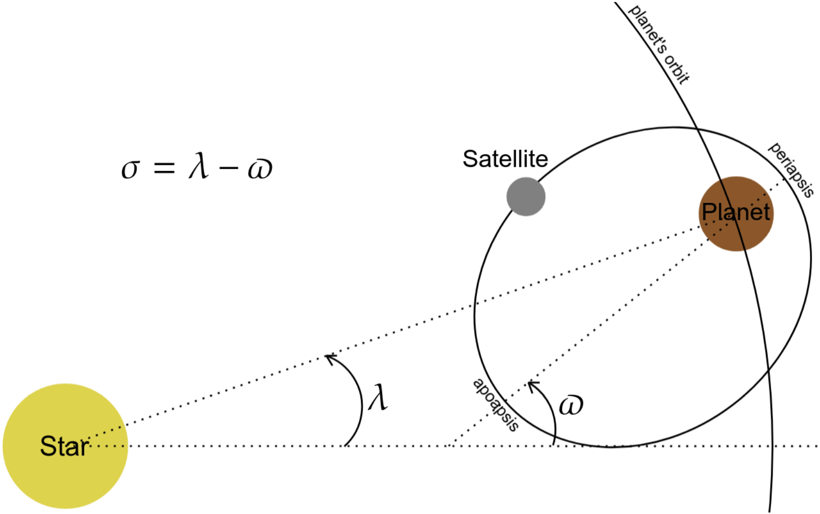
The evection resonance

$$\sigma = \lambda - \varpi$$



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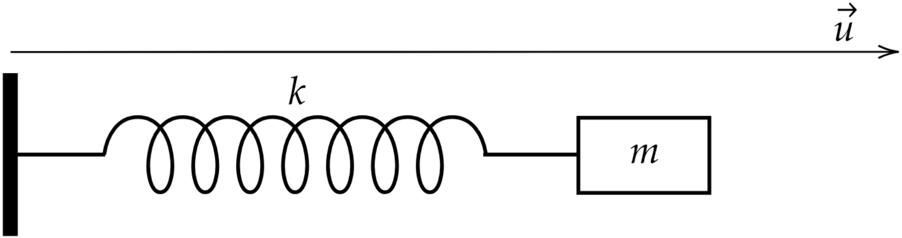
That looks too hard. Let's focus on the harmonic oscillator instead



$$\ddot{u} = -\frac{k}{m}u = -\omega_0^2 u, \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

To make things simpler, I choose k and m such that $\omega_0 = 1$

$$\boxed{\ddot{u} = -u}$$

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Let's put that in Hamiltonian form

$$\boxed{\ddot{u} = -u}$$

Let $v = \dot{u}$. The Hamiltonian $\mathcal{H}(u, v)$ must verify

$$\dot{u} = \frac{\partial \mathcal{H}}{\partial v}, \qquad \dot{v} = -\frac{\partial \mathcal{H}}{\partial u}$$

$$\rightarrow \boxed{\mathcal{H} = \frac{1}{2}\left(u^2 + v^2\right)}$$

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The Hamiltonian is conserved

$$\dot{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial u}\dot{u} + \frac{\partial \mathcal{H}}{\partial v}\dot{v} = -\dot{v}\dot{u} + \dot{u}\dot{v} = 0.$$

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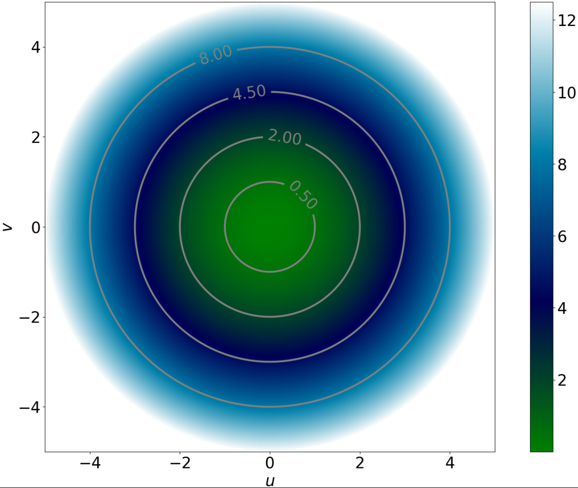
Phase space

$$\boxed{\mathcal{H} = \frac{1}{2}\left(u^2 + v^2\right)}$$



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Even simpler in polar coordinates (I, θ)

Previously:

$$\dot{u} = \frac{\partial \mathcal{H}}{\partial v}, \qquad \dot{v} = -\frac{\partial \mathcal{H}}{\partial u}$$

Now:

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial I}, \qquad \dot{I} = -\frac{\partial \mathcal{H}}{\partial \theta}$$

$$\rightarrow u = \sqrt{2I}\sin\theta, \quad v = \sqrt{2I}\cos\theta$$

$$\boxed{\mathcal{H} = I}$$

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial I} = 1, \quad \dot{I} = \frac{\partial \mathcal{H}}{\partial \theta} = 0$$

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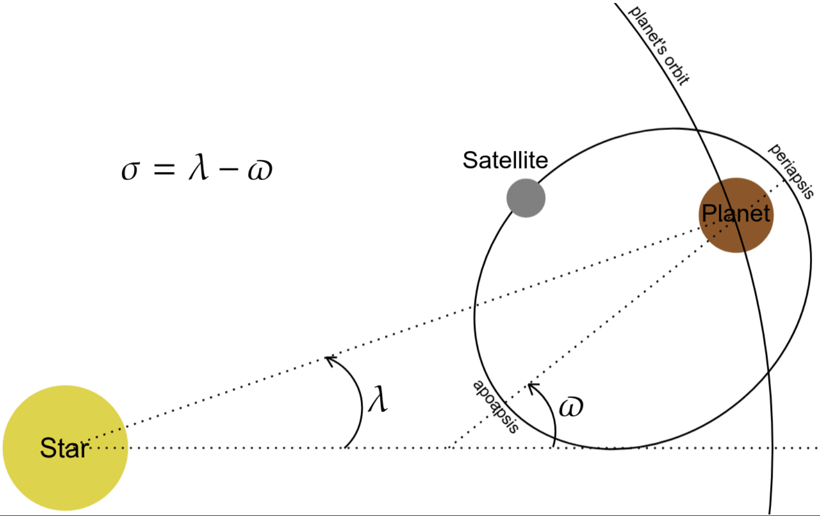
The exomoon is forced by λ

$$\sigma = \lambda - \varpi$$



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The exomoon is forced by the star: Let's force the harmonic oscillator

- Forcing on the exomoon : Angle $\boxed{\lambda = n^\star t}$ with $n^\star = \frac{2\pi}{P^\star}$
- Forcing on the harmonic oscillator : Defining $\boxed{\lambda = nt}$

Forced Harmonic oscillator (Mathieu differential equation)

$$\boxed{\ddot{u} = -u\,(1 + \varepsilon\cos\lambda)} \qquad \varepsilon \ll 1$$

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Forced Harmonic oscillator (Mathieu differential equation)

$$\boxed{\ddot{u} = -u\left(1 + \varepsilon\cos\lambda\right)} \qquad \varepsilon \ll 1$$

$$\boxed{\mathcal{H} = I + \frac{\varepsilon}{2} I \left(\cos\lambda - \frac{1}{2}\cos(2\theta - \lambda) - \frac{1}{2}\cos(2\theta + \lambda)\right)}$$

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Resonance $1:2$

$$\boxed{\mathcal{H} = I + \frac{\varepsilon}{2} I \left(\cos\lambda - \frac{1}{2}\cos(2\theta - \lambda) - \frac{1}{2}\cos(2\theta + \lambda) \right)}$$

Remember $\dot{\theta} = 1$ and $\lambda = nt$. Suppose $\boxed{n \approx 2}$.

Then

$$2\dot{\theta} - \dot{\lambda} = 2 - n \approx 0$$

$2\theta - \lambda$ is a slow angle whereas λ and $2\theta + \lambda$ are fast angles.

Let's average them out.

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Resonance $1:2$

$$\boxed{\mathcal{H} = I + \frac{\varepsilon}{2} I \left(\qquad - \frac{1}{2}\cos(2\theta - \lambda) \qquad \right)} + \mathcal{O}(\varepsilon^2)$$

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Resonance $1:2$

$$\boxed{\mathcal{H}(I,\theta,t) = I - \frac{\varepsilon}{4} I\cos(2\theta - \lambda)}$$

Let's remove the explicit time-dependency by viewing λ as a variable of the Hamiltonian.

We want $\dot{\lambda} = \dfrac{\partial \mathcal{H}}{\partial \Lambda} = n \;\rightarrow\;$ Adding $+n\Lambda$ to $\mathcal{H}$

$$\boxed{\mathcal{H}(I,\Lambda;\theta,\lambda) = I + n\Lambda - \frac{\varepsilon}{4} I\cos(2\theta - \lambda)}$$

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$$\dot{\theta} = \frac{\partial\mathcal{H}}{\partial I}, \quad \dot{\lambda} = \frac{\partial\mathcal{H}}{\partial\Lambda}, \quad \dot{I} = -\frac{\partial\mathcal{H}}{\partial\theta}, \quad \dot{\Lambda} = -\frac{\partial\mathcal{H}}{\partial\lambda}$$

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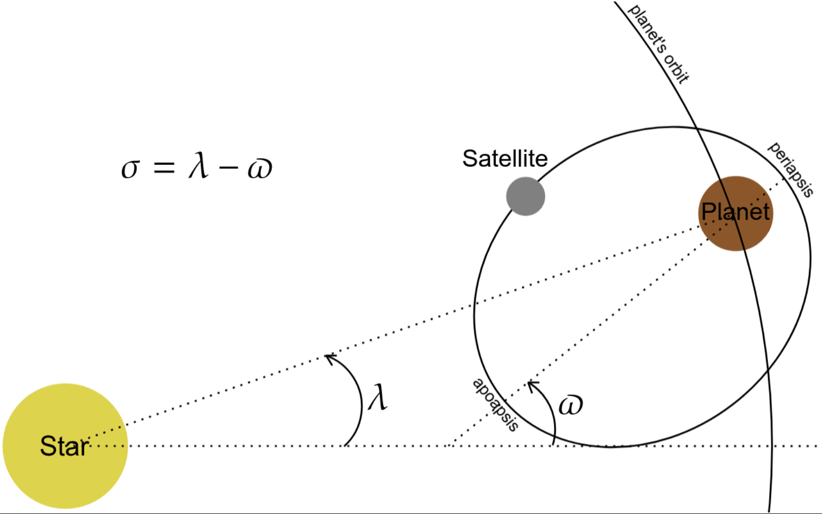
I need the resonant angle to appear explicitly

$$\sigma = \lambda - \varpi$$



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From two to one degree of freedom (Losing a pair of variables)

$$\boxed{\mathcal{H}(I, \Lambda; \theta, \lambda) = I + n\Lambda - \frac{\varepsilon}{4} I \cos(2\theta - \lambda)}$$

Resonant angle

- For the exomoon : $\boxed{\sigma = \varpi - \lambda}$ (resonance $1:1$).
- For the harmonic oscillator : $\boxed{\sigma = \theta - \lambda/2}$ (resonance $1:2$).

I also define $\sigma_2 = \lambda$, $\Sigma = I$, $\Sigma_2 = \Lambda + I/2$.

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$$\mathcal{H}(\Sigma,\Sigma_2;\sigma,\sigma_2) = \boxed{\left(1 - \frac{n}{2}\right)\Sigma + n\Sigma_2 - \frac{\varepsilon}{4}\Sigma\cos 2\sigma}$$

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$$\dot{\Sigma}_2 = \partial\mathcal{H}/\partial\sigma_2 = 0$$

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Going back to Cartesian coordinates

$$X = \sqrt{2\Sigma}\cos\sigma, \quad Y = \sqrt{2\Sigma}\sin\sigma$$

$$\boxed{\mathcal{H} = \frac{X^2}{2}\left(1 - \frac{n}{2} - \frac{\varepsilon}{4}\right) + \frac{Y^2}{2}\left(1 - \frac{n}{2} + \frac{\varepsilon}{4}\right)}$$

- Same signs $\rightarrow$ Level lines are ellipses $\rightarrow$ Stable
- Different signs $\rightarrow$ Level lines are hyperbolas $\rightarrow$ Unstable

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- Same signs $\rightarrow$ Level lines are ellipses $\rightarrow$ Stable
- Different signs $\rightarrow$ Level lines are hyperbolas $\rightarrow$ Unstable

Example : $n = 1.98 \rightarrow 1 - n/2 = 0.01$

- $\varepsilon < 0.04 \rightarrow$ Stable
- $\varepsilon > 0.04 \rightarrow$ Unstable

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Exomoon versus harmonic oscillator

- Harmonic oscillator

$$\mathcal{H} = \frac{X^2}{2}\left(1 - \frac{n}{2} - \frac{\varepsilon}{4}\right) + \frac{Y^2}{2}\left(1 - \frac{n}{2} + \frac{\varepsilon}{4}\right)$$

- Exomoon

$$\mathcal{H} = \frac{X^2}{2}\left(1 - \frac{9}{2}\frac{P}{P^\star}\right) + \frac{Y^2}{2}\left(1 + 3\frac{P}{P^\star}\right)$$

With $X \propto e\cos\sigma$ and $Y \propto e\sin\sigma$

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Exomoon versus harmonic oscillator

- Harmonic oscillator

$$\mathcal{H} = \frac{X^2}{2}\left(1 - \frac{n}{2} - \frac{\varepsilon}{4}\right) + \frac{Y^2}{2}\left(1 - \frac{n}{2} + \frac{\varepsilon}{4}\right)$$

- Exomoon

$$\mathcal{H} = \frac{X^2}{2}\left(1 - \frac{9}{2}\frac{P}{P^\star}\right) + \frac{Y^2}{2}\left(1 + 3\frac{P}{P^\star}\right)$$

With $X \propto e\cos\sigma$ and $Y \propto e\sin\sigma$

$\frac{P}{P^\star} > \frac{2}{9} \ \leftrightarrow \ a > 0.53\, R_{\text{Hill}} \ \leftrightarrow$ Exomoon's eccentricity blows up

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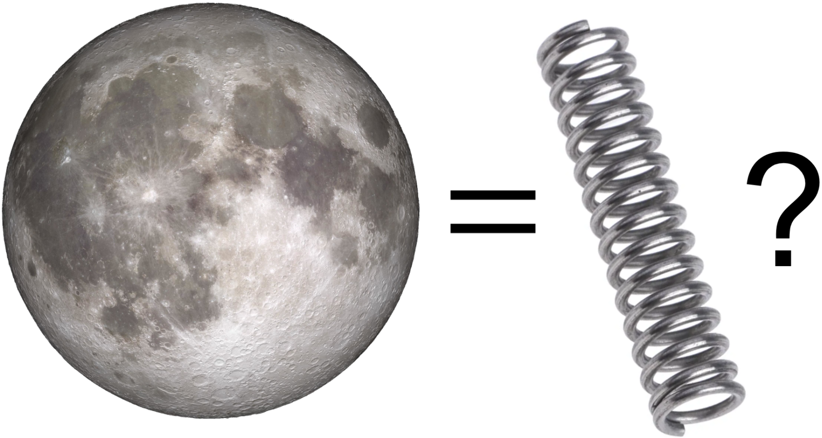
Indeed !



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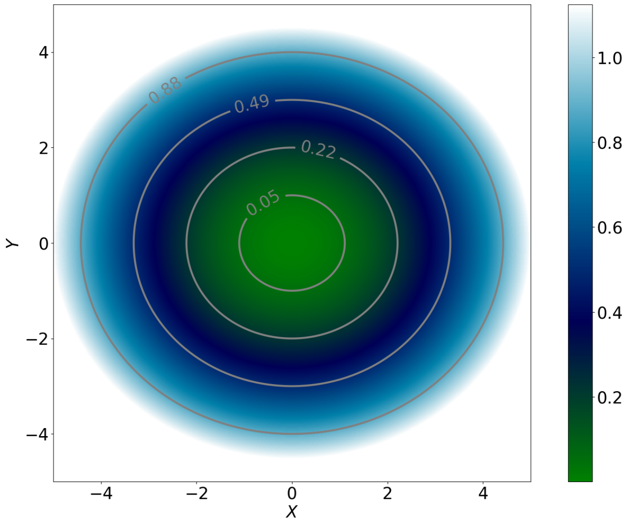
$\varepsilon = 0.04,\ n = 1.8$



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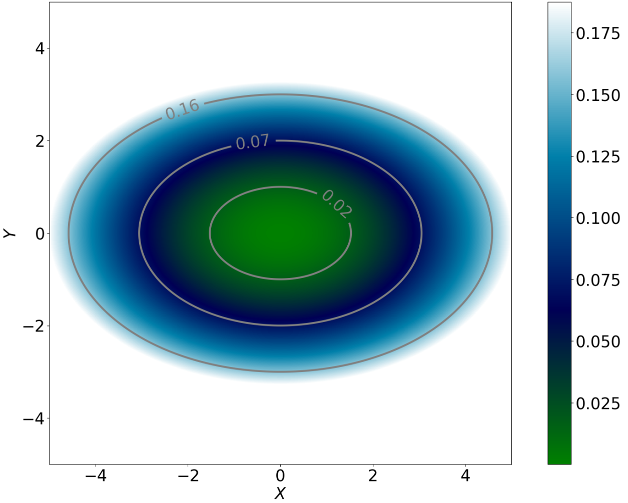
$\varepsilon = 0.04,\ n = 1.95$



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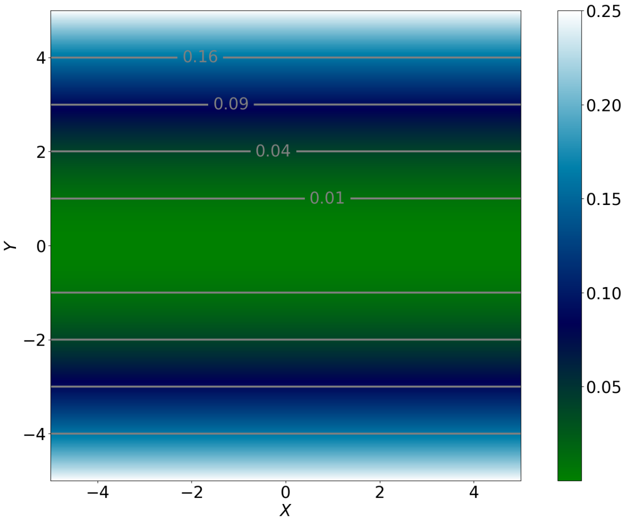
$\varepsilon = 0.04,\ n = 1.98$



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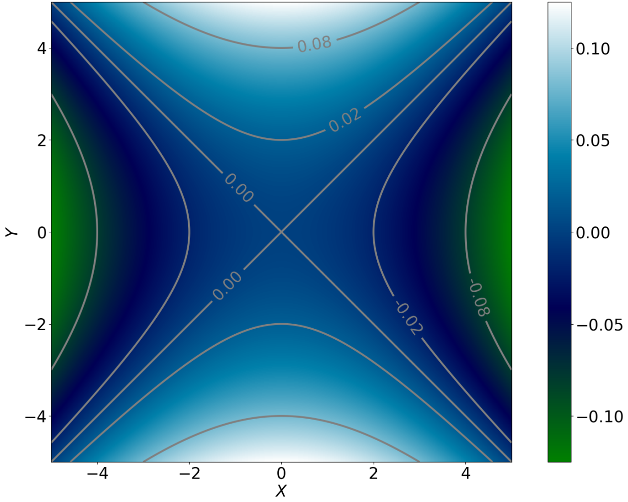
$\varepsilon = 0.04, \; n = 2$



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$$R_{\mathsf{Hill}} = a^\star \left(\frac{m_{\mathsf{P}}}{3m^\star}\right)^{1/3}$$
$$R_{\mathsf{Roche}} \approx 2.4 R_{\mathsf{P}}$$

For the Sun-Earth-Moon system

$$0.53 R_{\mathsf{Hill}} = 124 R_\oplus, \quad R_{\mathsf{Roche}} = 2.4 R_\oplus$$

There is plenty of space for the Moon.

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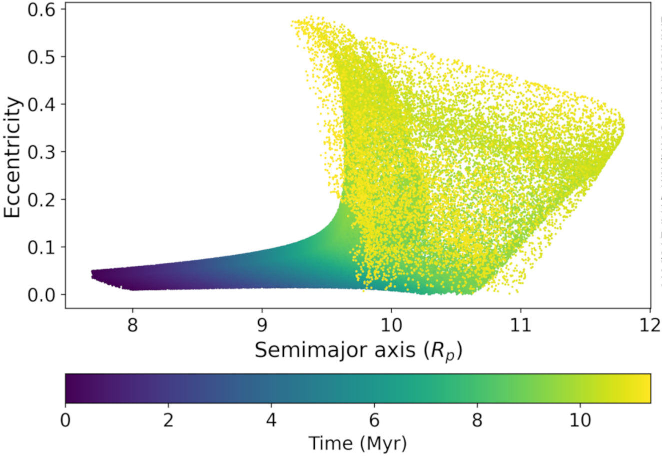
TOI-6303 b (With an hypothetical satellite)

$$0.47R_{\mathrm{Hill}} \approx 11R_{\mathrm{P}}, \quad R_{\mathrm{Roche}} \approx 2.4R_{\mathrm{P}}$$



Extracted from Sonanan Oi APL report

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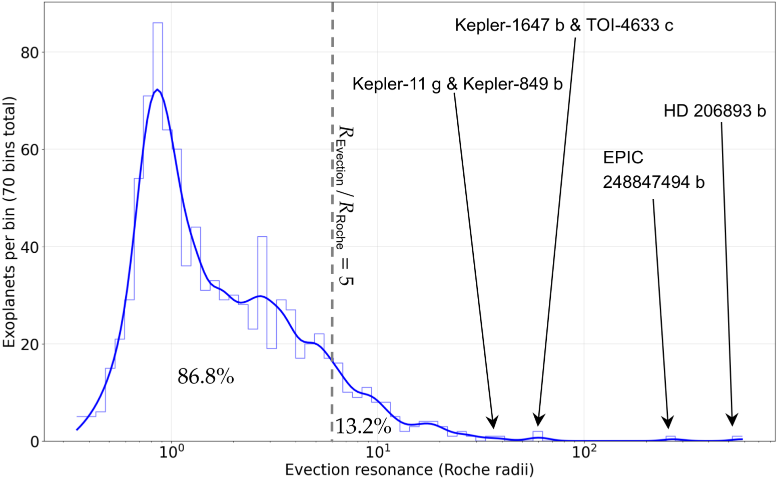
Evection distance for some exoplanets (956 featured exoplanets)

Exoplanets must have (a, P, m, R) constrained and $e < 0.2$ to be featured



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Exoplanets for which $R_{\text{Evection}}/R_{\text{Roche}} > 20$

Exoplanet	$\frac{R_{\text{Evection}}}{R_{\text{Roche}}}$	Exoplanet	$\frac{R_{\text{Evection}}}{R_{\text{Roche}}}$
HD 206893 b	581	Kepler-111 c	28.0
EPIC 248847494 b	271	Kepler-47 b	26.8
Kepler-1647 b	62.6	TIC 172900988 b	25.5
TOI-4633 c	57.8	Kepler-38 b	22.8
Kepler-849 b	39.4	PH1 b	21.1
Kepler-11 g	33.3	TIC 4672985 b	20.7
Mercury	17.7	Jupiter	149
Venus	32.8	Saturn	220
Earth	46.1	Uranus	542
Mars	62.7	Nepture	929

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How long would an exomoon typically survive

- An exomoon on a circular orbit at $a = \left(R_{\text{Roche}} + R_{\text{Evection}}\right)/2$
- $M_{\text{Exomoon}} = M_{\text{P}}/200$
- $k_2/Q = 10^{-4}$ if gas giant, 10^{-3} if Earth, 10^{-2} if super-Earth

Migration rate

$$\frac{\dot{a}}{a} = \frac{3k_2}{Q}\frac{M_{\text{Exomoon}}}{M_{\text{P}}}\left(\frac{R_{\text{P}}}{a}\right)^5 n$$

- I compute the time until the evection resonance is reached

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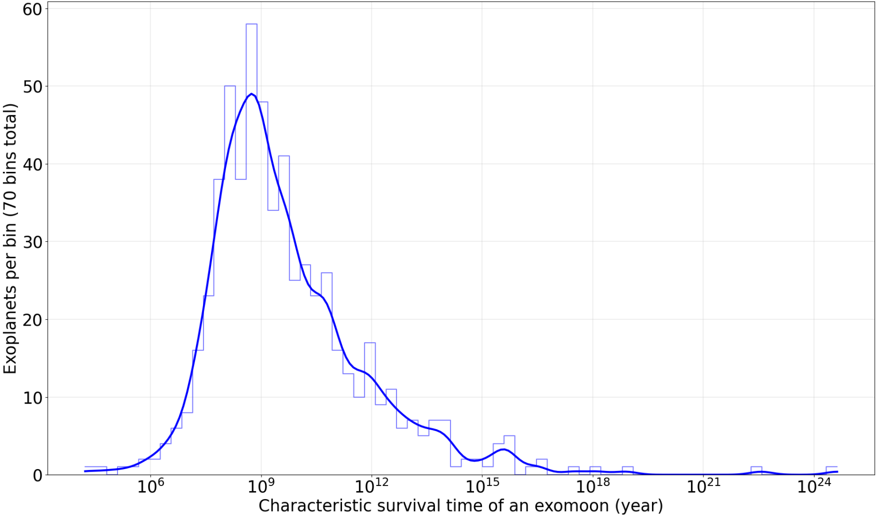
Characteristic survival time



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Candidates exoplanets

- Survival time $> 13.7 \times 10^9$ years $\rightarrow$ Can retain the exomoon
- $R_{\text{Evection}}/R_{\text{Roche}} > 10$ $\rightarrow$ Can acquire the exomoon

BD+20 594 b, CoRoT-9 b, EPIC 248847494 b, GJ 143 b, HD 136352 d, HD 206893 b, HD 219134 d, HD 219134 f, HD 95338 b, K2-263 b, K2-3 d, Kepler-10 c, Kepler-102 f, Kepler-103 c, Kepler-11 g, Kepler-111 c, Kepler-16 b, Kepler-1647 b, Kepler-1661 b, Kepler-34 b, Kepler-37 b, Kepler-38 b, Kepler-413 b, Kepler-453 b, Kepler-46 b, Kepler-47 b, Kepler-47 d, Kepler-538 b, Kepler-849 b, Kepler-87 b, PH1 b, TIC 172900988 b, TIC 4672985 b, TOI-2088 b, TOI-2095 c, TOI-2529 b, TOI-4504 c, TOI-4633 c, TOI-5542 b, TOI-561 e

Thank you for your attention