Analytical expression of tides in the constant-Δt model with Poincaré rectangular coordinates

Jérémy COUTURIER

August 26, 2025

Contents

Abstract

In this document, I use a pseudo-Hamiltonian formalism is order to obtain an analytical expression of the differential system of equations due to tides in the constant-Δt model. I use Poincaré's rectangular coordinates and calculations are performed with the algebraic manipulator *celeries* available as a python package. The setup is a coplanar planetary system with N planets and I only consider tides raised on the planets by the star and interacted with by the star.

1 Tidal model

I consider a planetary system of N planets of masses m_j and radii R_j orbiting a star of mass m_0. In this work, the Latin letter i is an index, while the Greek letter ι is $\sqrt{-1}$. Planets are assumed to be extended bodies subject to tides. Let $\boldsymbol{r}_j$ be the position of the star with respect to the j^{th} planet in the frame $(\mathcal{O}, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K})$ attached to the planet's rotation. Let $\boldsymbol{r}$ be a point within the planet in that same reference frame. The star raises at $\boldsymbol{r}$ the potential per unit mass

$$W'(\boldsymbol{r}) = -\frac{\mathcal{G}m_0}{|\boldsymbol{r}_j - \boldsymbol{r}|} = -\frac{\mathcal{G}m_0}{r_j}\sum_{l=0}^{+\infty}\left(\frac{r}{r_j}\right)^l P_l\left(\frac{\boldsymbol{r}\cdot\boldsymbol{r}_j}{rr_j}\right), \tag{1}$$

called perturbing potential. In this expression, the P_l are the Legendre polynomials. Terms $l \leq 1$ have a gradient independent on $\boldsymbol{r}$. Since tides come from a differential acceleration within the planet, these terms do not contribute to tides and are discarded.

Terms $l \geq 3$ are also discarded due to the small size of r/r_j. Using $P_2(z) = (3z^2 - 1)/2$, the perturbing potential is therefore rewritten

$$W(\boldsymbol{r}) = -\frac{1}{2}\frac{\mathcal{G}m_0 r^2}{r_j^3}\left(3\frac{(\boldsymbol{r}\cdot\boldsymbol{r}_j)^2}{r^2 r_j^2} - 1\right). \tag{2}$$

The perturbing potential W governs how the star redistributes mass in the planet. The redistribution of mass raises a potential V, called perturbed potential. The star itself or any other body feels the potential V and is thus affected by tides. The challenge in tidal theories is to obtain the perturbed potential V from the perturbing potential W. Assuming these three hypothesis

(i) The tidal perturbations are small enough so that V depends linearly on W,

(ii) The geophysical properties of planet j are constant over the tidal timescales,

(iii) Planet j is isotropic in the absence of tides,

it can be shown (Boué *et al.*, 2019; Couturier, 2022) that, at the quadrupolar order, V is related to W by the convolution product

$$V(\boldsymbol{r},t) = \frac{R_j^3}{r^3}\int_{-\infty}^{t} k_2^{(j)}(t-t')W\left(R_j\frac{\boldsymbol{r}}{r},t'\right)dt', \tag{3}$$

where $k_2^{(j)}(t)$ is the second Love distribution of planet j and characterizes its memory of past stresses. In order to get rid of the convolution product, a common choice is the constant-Δt model where $k_2^{(j)}(t)$ is taken proportional to a Dirac distribution

$$k_2^{(j)}(t) = \kappa_2^{(j)}\,\delta(t - \Delta t^{(j)}). \tag{4}$$

In this expression, the second Love number $\kappa_2^{(j)}$ measures the ability of planet j to get deformed by tides and the timelag $\Delta t^{(j)}$ measures its ability to dissipate energy when it is deformed by tides. Injecting Eqs. (2) and (4) into Eq. (3), I obtain for the perturbed potential by unit mass felt by a body at position $\boldsymbol{r}_k$

$$V(\boldsymbol{r}_k) = -\frac{1}{2}\kappa_2^{(j)}\frac{\mathcal{G}m_0 R_j^5}{r_k^5 r_j^{\star 5}}\left[3\left(\boldsymbol{r}_k\cdot\boldsymbol{r}_j^\star\right)^2 - r_k^2 r_j^{\star 2}\right], \tag{5}$$

where, in the rotating frame, $\boldsymbol{r}_j^\star = \boldsymbol{r}_j(t - \Delta t^{(j)})$. From now on, I denote $\varsigma^\star = \varsigma(t - \Delta t^{(j)})$ for any quantity ς. The full potential felt by a body of mass m_k at $\boldsymbol{r}_k$ is $m_k V(\boldsymbol{r}_k)$. Either the star or any other planet can play the role of body k. Because the planet's masses are assumed much smaller than the star's mass, I disregard cases where body k is a planet and I only consider tides interacted with by the star, leading to $\boldsymbol{r}_k = \boldsymbol{r}_j$. Calling $S = w_j - w_j^\star$ the angle between $\boldsymbol{r}_j$ and $\boldsymbol{r}_j^\star$, with w_j the true longitude of planet j, the perturbed potential is rewritten

$$V(\boldsymbol{r}_j) = -\frac{1}{2}\kappa_2^{(j)}\frac{\mathcal{G}m_0 R_j^5}{r_j^3 r_j^{\star 3}}\left(3\cos^2 S - 1\right). \tag{6}$$

The perturbed potential is currently written in the frame attached to the planet's rotation. I switch to the inertial reference frame $(\mathcal{O}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ by writing

$$S = w_j - w_j^\star - \left(\theta_j - \theta_j^\star\right), \tag{7}$$

where θ_j is the sideral rotation angle of planet j.

2 Pseudo-Hamiltonian formalism

Tides can be included in a planetary system by adding the potential $H_t^{(j)} = m_0 V(\boldsymbol{r}_j)$ to the Hamiltonian of the problem. The resulting scalar field is *pseudo*-Hamiltonian because it depends on past times (in that case on time $t - \Delta t^{(j)}$). However, the equations of motions with tidal dissipation can still be obtained from the Hamilton equations if the starred variables are kept constant while differentiating (because $\boldsymbol{r}_j^\star$ is not a variable in Eq. (5), it is a time-dependent forcing).

However, unlike a regular Hamiltonian, this *pseudo*-Hamiltonian is merely a tool used to obtain the contributions due to tides in the equations of motion. Because problems of celestial mechanics are often dealt with efficiently using Poincaré's canonical rectangular coordinates $(\Lambda_j, x_j; \lambda_j, -\imath\bar{x}_j)$, I use these coordinates from now on, and more specifically their non-canonical version $X_j = \sqrt{2/\Lambda_j}\, x_j$. A definition of these variables can be found in Sect. 1.1. of this work[1]. By conservation of the total angular momentum, the sideral rotation θ_j of planet j is also a variable of the problem. Its conjugated action is $\Theta_j = \alpha_j m_j R_j^2 \dot{\theta}_j$, where $\alpha_j m_j R_j^2$ is the principal moment of inertia of planet j and α_j is a dimensionless structure constant equal to $2/5$ for an homogeneous planet. The kinetic energy T_j of rotation of the planet is

$$T_j = \frac{\Theta_j^2}{2\alpha_j m_j R_j^2}, \tag{8}$$

and I add it to the Hamiltonian. In order to work with dimensionless variables, I define

$$\mathcal{L}_j = \frac{\Lambda_j}{\Lambda_{j,0}} \approx 1, \quad \tilde{\Theta}_j = \frac{\Theta_j}{\Lambda_{j,0}}, \quad X_j' = \sqrt{\frac{2}{\Lambda_{j,0}}} x_j = \mathcal{L}_j^{1/2} X_j, \tag{9}$$

where $\Lambda_{j,0}$ is a nominal value for Λ_j, such that $\mathcal{L}_j \approx 1$. In order to stay as close as possible from the regular Hamilton equations, I renormalize the Hamiltonian with

$$\mathcal{H}_t^{(j)} = \frac{H_t^{(j)}}{\Lambda_{j,0}}, \quad \mathcal{T}_j = \frac{T_j}{\Lambda_{j,0}}. \tag{10}$$

Writing $\mathcal{H}_j = \mathcal{H}_t^{(j)} + \mathcal{T}_j$, the tidal equations of motion are

$$\dot{\mathcal{L}}_j = -\frac{\partial \mathcal{H}_j}{\partial \lambda_j}, \quad \dot{\lambda}_j = \frac{\partial \mathcal{H}_j}{\partial \mathcal{L}_j}, \quad \dot{X}_j' = -2\imath \frac{\partial \mathcal{H}_j}{\partial \bar{X}_j'}, \quad \dot{\tilde{\Theta}}_j = -\frac{\partial \mathcal{H}_j}{\partial \theta_j}, \quad \dot{\theta}_j = \frac{\partial \mathcal{H}_j}{\partial \tilde{\Theta}_j}, \tag{11}$$

all these partial derivatives being computed while keeping the starred quantities constant. The expression of $\mathcal{H}_t^{(j)}$ averaged over the mean motion of planet j is

$$\mathcal{H}_t^{(j)} = -n_{j,0} q_j \frac{m_0}{m_j} \mathcal{L}_j^{-6} \mathcal{L}_j^{\star -6} \left(\frac{1}{4} + \frac{3}{4} \cos 2(\lambda_j - \lambda_j^\star - \theta_j + \theta_j^\star) + \sum_{k=1}^{+\infty} \Xi_{2k}^{(j)} \right), \tag{12}$$

where $q_j = \kappa_2^{(j)} R_j^5 / a_{j,0}^5$. The coefficients $\Xi_{2k}^{(j)}$ of the expansion in eccentricity are given by (Couturier *et al.*, 2021, Eq. (42) and Appendix B) up to $k = 2$ (beware that $\mathcal{R}_j$ and X_j are written in this work instead of $\mathcal{L}_j$ and X_j').

[1] https://jeremycouturier.com/3pla/SecondOrderMass.pdf

3 Contribution of tides to the equations of motions

Once the equations of motions are obtained from the Hamilton equations, the starred variables can be given as a function of the non-starred variables. Because $\Delta_t^{(j)}$ is much smaller than the secular timescales, I simply make the substitution $\mathcal{L}_j^\star = \mathcal{L}_j$ and $X_j'^\star = X_j'$. For λ_j and θ_j, I define $Q_j^{-1} = n_{j,0}\Delta_t^{(j)}$ and $\omega_j = \dot{\theta}_j/n_{j,0}$ and a first-order expansion in $\Delta_t^{(j)}$ yields

$$\begin{aligned}
\lambda_j - \lambda_j^\star &= n_j\Delta_t^{(j)} = \frac{n_j}{n_{j,0}} n_{j,0}\Delta_t^{(j)} = \mathcal{L}_j^{-3} Q_j^{-1}, \\
\theta_j - \theta_j^\star &= \dot{\theta}_j \Delta_t^{(j)} = \omega_j Q_j^{-1}.
\end{aligned} \tag{13}$$

I compute the equations of motion and obtain, for the regular Poincaré coordinates

$$\begin{aligned}
\dot{X}_j &= -3n_{j,0}\frac{q_j}{Q_j}\frac{m_0}{m_j}\mathcal{L}_j^{-13}X_j\left(\sum_{n=1}^{+\infty}\left(p_{2n}^{(j)} - p_{2n}\iota Q_j\right)X_j^{n-1}\bar{X}_j^{n-1}\right), \\
\dot{\lambda}_j &= 6n_{j,0}q_j\frac{m_0}{m_j}\mathcal{L}_j^{-13}\left(1 + \sum_{n=1}^{+\infty} q_{2n}X_j^n\bar{X}_j^n\right), \\
\dot{\Lambda}_j &= -3n_{j,0}\Lambda_{j,0}\frac{q_j}{Q_j}\frac{m_0}{m_j}\mathcal{L}_j^{-12}\left(\mathcal{L}_j^{-3} - \omega_j + \sum_{n=1}^{+\infty} k_{2n}^{(j)}X_j^n\bar{X}_j^n\right), \\
\ddot{\theta}_j &= 3n_{j,0}^2\alpha_j^{-1}\frac{q_j}{Q_j}\frac{a_{j,0}^2}{R_j^2}\frac{m_0}{m_j}\mathcal{L}_j^{-12}\left(\mathcal{L}_j^{-3} - \omega_j + \sum_{n=1}^{+\infty} h_{2n}^{(j)}X_j^n\bar{X}_j^n\right),
\end{aligned} \tag{14}$$

where I recall that $q_j = \kappa_2^{(j)} R_j^5/a_{j,0}^5$ and

$$\begin{aligned}
p_{2n}^{(j)} &= p_{2n}'\mathcal{L}_j^{-3} - p_{2n}''\omega_j, \\
k_{2n}^{(j)} &= k_{2n}'\mathcal{L}_j^{-3} - k_{2n}''\omega_j, \\
h_{2n}^{(j)} &= h_{2n}'\mathcal{L}_j^{-3} - h_{2n}''\omega_j,
\end{aligned} \tag{15}$$

Using the algebraic manipulator *celeries*, I computed the equations of motions up to order 9 in eccentricity and found the coefficients

$$\begin{aligned}
(q_0, q_2, q_4, q_6, q_8) &= \left(1, \frac{65}{8}, \frac{455}{16}, \frac{525}{8}, \frac{5\,750\,747}{49\,152}\right), \\
(p_0, p_2, p_4, p_6, p_8, p_{10}) &= \left(0, \frac{5}{2}, \frac{65}{4}, \frac{105}{2}, \frac{3745}{32}, \frac{52605}{256}\right), \\
(p_0', p_2', p_4', p_6', p_8', p_{10}') &= \left(0, \frac{19}{2}, 106, \frac{4351}{8}, \frac{58\,395}{32}, \frac{1\,185\,825}{256}\right), \\
(p_0'', p_2'', p_4'', p_6'', p_8'', p_{10}'') &= \left(0, 6, \frac{351}{8}, \frac{637}{4}, \frac{12\,705}{32}, \frac{12\,411}{16}\right), \\
(k_0', k_2', k_4', k_6', k_8'') &= \left(1, 23, \frac{697}{4}, \frac{6045}{8}, \frac{170\,284\,187}{73\,728}\right), \\
(k_0'', k_2'', k_4'', k_6'', k_8'') = (h_0', h_2', h_4', h_6', h_8') &= \left(1, \frac{27}{2}, \frac{273}{4}, \frac{847}{4}, \frac{35\,742\,107}{73\,728}\right), \\
(h_0'', h_2'', h_4'', h_6'', h_8'') &= \left(1, \frac{15}{2}, \frac{195}{8}, \frac{105}{2}, \frac{6\,469\,787}{73\,728}\right).
\end{aligned} \tag{16}$$

The conservation of the total angular momentum of the system reads (Couturier *et al.*, 2021, Appendix D)

$$h_{2n}^{(j)} - k_{2n}^{(j)} + p_{2n}^{(j)} = 0 \quad \forall n \in \mathbb{N}, \tag{17}$$

and the differential system of equations (14) does indeed preserve the total angular momentum at all orders in eccentricity.

The coefficients p_{2n} and q_{2n} do not intervene in the equation of conservation of the total angular momentum because they correspond to elastic tides[2] and they do not affect the angular momentum of the system. These coefficients can safely be evaluated to zero in Eqs. (14) when only the dissipative contributions of tides matter (*e.g.* when computing the real parts of the eigenvalues around the equilibria). When studying a mean-motion resonance, the differential system (14) can be evaluated at the keplerian resonance $\Lambda_j = \Lambda_{j,0}$ by substituting $\mathcal{L}_j = 1$, but this kills all the dissipation in the libration amplitude of the resonance angle and is generally not a viable approximation.

Sometimes, it is more useful to know the quantities $\dot{D}_j$ and $\dot{\varpi}_j$ instead of $\dot{X}_j$, where $X_j = \sqrt{2D_j/\Lambda_j e^{\iota\varpi_j}}$ and I recall that $X'_j = \sqrt{2D_j/\Lambda_{j,0}}\, e^{\iota\varpi_j} = \mathcal{L}_j^{1/2} X_j$. Writing

$$\dot{X}'_j = X'_j \left(\frac{1}{2}\frac{\dot{D}_j}{D_j} + \iota\dot{\varpi}_j \right) = \frac{1}{2}\dot{\mathcal{L}}_j \mathcal{L}_j^{-1/2} X_j + \mathcal{L}_j^{1/2}\dot{X}_j, \tag{18}$$

I get

$$\begin{aligned} \dot{D}_j &= -3n_{j,0}\frac{q_j}{Q_j}\frac{m_0}{m_j}\mathcal{L}_j^{-13} D_j \left(20\mathcal{L}_j^{-3} - 13\omega_j + \sum_{n=1}^{+\infty} \left(k_{2n}^{(j)} + 2p_{2n+2}^{(j)} \right) D_j^n \right), \\ \dot{\varpi}_j &= 3n_{j,0} q_j \frac{m_0}{m_j}\mathcal{L}_j^{-13} \left(\frac{5}{2} + \sum_{n=1}^{+\infty} p_{2n+2} D_j^n \right). \end{aligned} \tag{19}$$

References

Boué, G., Correia, A. C. M. and Laskar, J. (2019). On Tidal Theories and the Rotation of Viscous Bodies. 82, pp. 91–98.

Couturier, J. (2022). *Dynamics of Co-Orbital Planets. Tides and Resonance Chains.* PhD thesis. Observatoire de Paris, https://theses.hal.science/tel-04197740.

Couturier, J., Robutel, P. and Correia, A. C. M. (2021). An Analytical Model for Tidal Evolution in Co-Orbital Systems. I. Application to Exoplanets. *Celestial Mechanics and Dynamical Astronomy*, 133.8, p. 37.

[2] They are the terms of Eqs. (14) independent on Q_j, for which $\ddot{\boldsymbol{r}}_j \propto \boldsymbol{r}_j$