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Points fixes et centres de libration
dans les chaînes de résonance

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Séminaire ASD

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Exoplanetary systems often are in resonance chains

- Examples: Kepler-223 $(3:4:6:8)$ and TOI-178 $(0:2:4:6:9:12)$.
- In the solar system: The four galilean satellites $(1:2:4:0)$.
- A generic n-planets chain is denoted $p_1 : p_2 : \cdots : p_{n-1} : p_n$.

Definition

- A pair (i,j) of planet is resonant if $T_i/T_j \sim p_i/p_j \in \mathbb{Q}$.
- Or equivalently (Kepler third law) if $a_i/a_j \sim (p_i/p_j)^{2/3}$.

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Two planets in resonance $p : p+1$ (planar case)

$$\boxed{\frac{T_2}{T_1} \approx \frac{p+1}{p}}$$

e.g. $T_2/T_1 \approx 3/2$ for Kepler-1972

- We use Poincaré's canonical variables

$$\Lambda_j = \beta_j \sqrt{\mu_j a_j} \quad \leftrightarrow \quad \lambda = \text{mean longitude},$$

$$D_j = \Lambda_j \left(1 - \sqrt{1 - e_j^2}\right) \quad \leftrightarrow \quad -\varpi_j = -\text{longitude of the periapsis},$$

where $\beta_j = m_0 m_j / \left(m_0 + m_j\right) \approx m_j$ and $\mu_j = \mathcal{G}\sqrt{m_0 + m_j}$.

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Hamiltonian

$$\mathcal{H} = \mathcal{H}_K(\Lambda_1,\Lambda_2) + \mathcal{H}_P(\Lambda_1,\Lambda_2,D_1,D_2;\lambda_1,\lambda_2,-\varpi_1,-\varpi_2),$$

$$\mathcal{H}_K(\Lambda_1,\Lambda_2) = -\sum_{j=1}^{2}\frac{\beta_j^3\mu_j^2}{2\Lambda_j^2}.$$

- The perturbation is expanded in a Fourier serie of eccentricities

$$\mathcal{H}_P = \sum_{\mathbf{k}\in\mathbb{Z}^2}\left(\sum_{\mathbf{q}\in\mathbb{N}^4}\Xi(a_1/a_2)X_1^{q_1}X_2^{q_2}\bar{X}_1^{q_3}\bar{X}_2^{q_4}\right)e^{i(k_1\lambda_1+k_2\lambda_2)},$$

where $X_j = e_j\exp i\varpi_j + \mathcal{O}(e_j^3)$.

Invariance by rotation (D'Alembert rule):

$$k_1+k_2+q_1+q_2-q_3-q_4=0$$

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Averaging and truncation

- Most of the angles $k_1\lambda_1 + k_2\lambda_2$ are fast-circulating.
- We only keep the slow ones
- Example: Resonance $2 : 3 \rightarrow$ We keep $2\lambda_1 - 3\lambda_2$ and the multiples.
- Order r in eccentricity: We only keep $|q_1| + |q_2| + |q_3| + |q_4| \leq r$

First order in eccentricity in resonance $p : p + 1$

$$\mathcal{H}_P = \frac{\mathcal{G}m_1m_2}{a_2}\left[f_1\sqrt{\frac{2D_1}{\Lambda_1}}\cos(p\lambda_1 - (p+1)\,\lambda_2 + \varpi_1)\right. \\ \left. + f_2\sqrt{\frac{2D_2}{\Lambda_2}}\cos(p\lambda_1 - (p+1)\,\lambda_2 + \varpi_2)\right].$$

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Reduction to one degree of freedom

- Two angles of interest $\sigma_j = p\lambda_1 - (p+1)\,\lambda_2 + \varpi_j$
- Transformation $(e_1, e_2; \sigma_1, \sigma_2) \to (R, S; r, s)$
- s absent from the Hamiltonian $\to S$ is a conserved quantity.

Conserved quantities

- Total angular momentum $G = m_1\sqrt{\mu a_1\left(1-e_1^2\right)} + m_2\sqrt{\mu a_2\left(1-e_2^2\right)}$
- Scaling parameter $\Gamma = (p+1)\,m_1\sqrt{\mu a_1} + pm_2\sqrt{\mu a_2}$
- $S \propto \Upsilon = \dfrac{e_{1,\mathrm{eq}}}{e_{2,\mathrm{eq}}}e_2^2 + \dfrac{e_{2,\mathrm{eq}}}{e_{1,\mathrm{eq}}}e_1^2 + 2e_1e_2\cos\left(\varpi_1 - \varpi_2\right)$

e.g. $\Upsilon_{2:3} = 1.07148\dfrac{m_1}{m_2}e_1^2 + 0.93329\dfrac{m_2}{m_1}e_2^2 + 2e_1e_2\cos\left(\varpi_1 - \varpi_2\right)$

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Reduction to one parameter

$$\mathcal{H} = \alpha R - \beta R^2 + \gamma\sqrt{2R}\cos r$$

$$\beta(p, m_1, m_2), \quad \gamma(p, m_1, m_2), \quad \alpha(p, m_1, m_2, G+S)$$

- Rescaling action and time : $\Sigma = (2\beta/\gamma)^{2/3} R$ and $\tau = (\beta\gamma^2/4)^{1/3} t$

$$\boxed{\mathcal{H} = 3\delta\Sigma - \Sigma^2 + 2\sqrt{2\Sigma}\cos\sigma} \quad \delta = \alpha\left(\frac{4}{27\beta\gamma^2}\right)^{1/3}$$

- Using cartesian coordinates $X = \sqrt{2\Sigma}\cos\sigma$ and $Y = \sqrt{2\Sigma}\sin\sigma$

$$\boxed{\mathcal{H} = 2X - \frac{1}{4}\left(X^2 + Y^2 - 3\delta\right)^2}$$

- $|X| \sim \mathcal{O}\left(e_j\left(\frac{m_0}{m_j}\right)^{1/3}\right)$ and $|Y| \sim \mathcal{O}\left(e_j\left(\frac{m_0}{m_j}\right)^{1/3}\right)$

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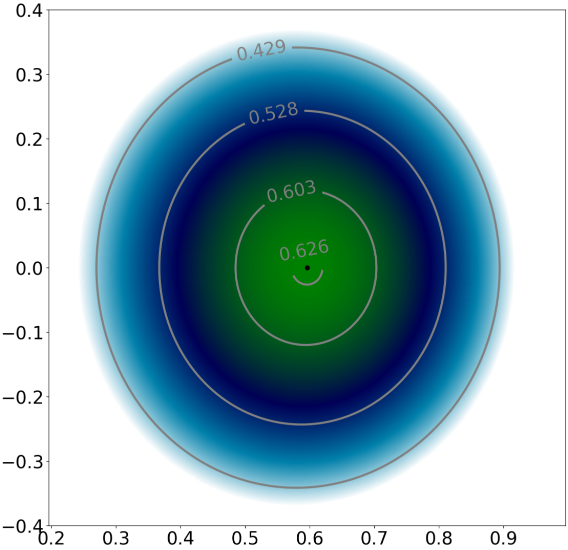
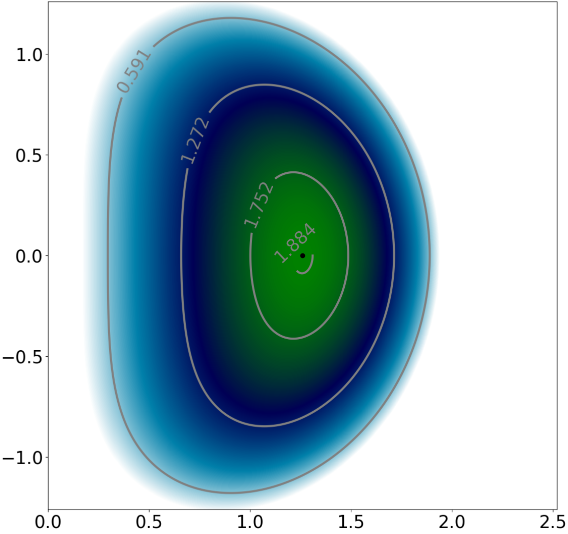
$\delta = -1$



$$\delta = 0$$



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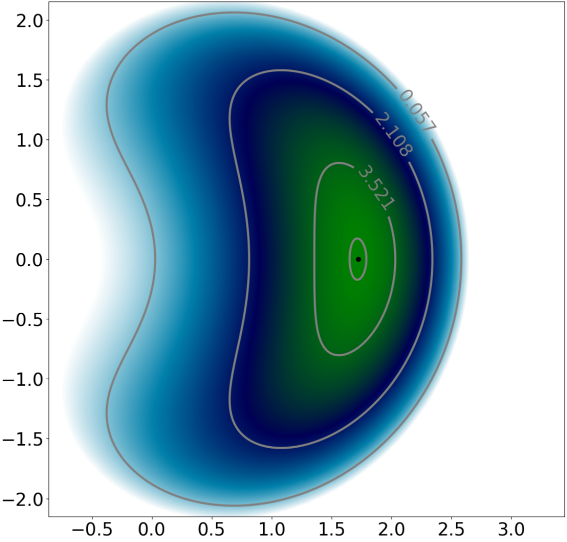
$\delta = 0.6$



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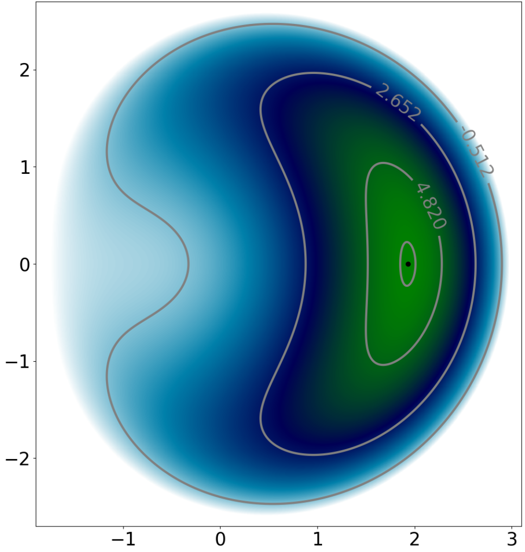
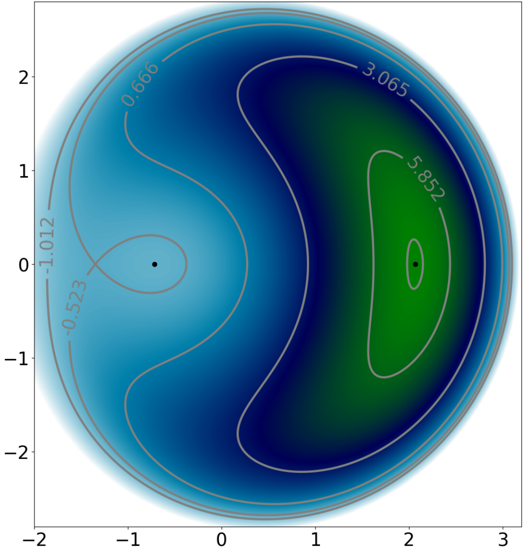
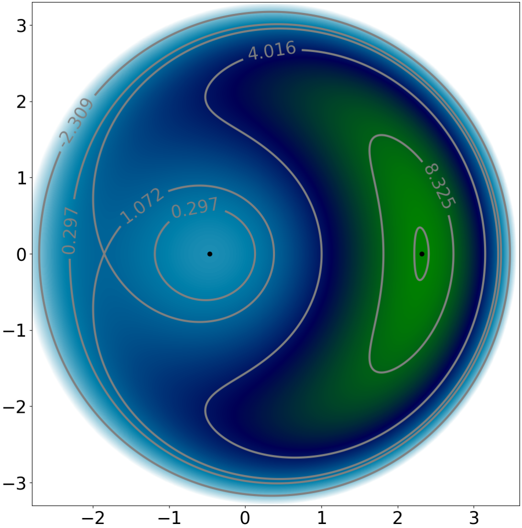
$\delta = 0.9$



$\delta = 1.1$



$$\delta = 1.5$$



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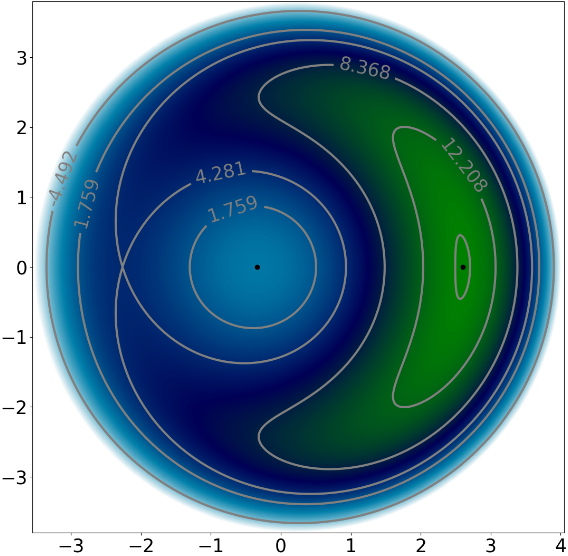
$\delta = 2$



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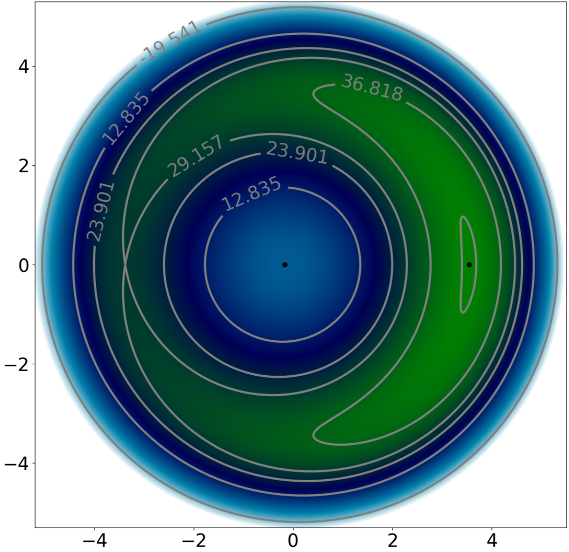
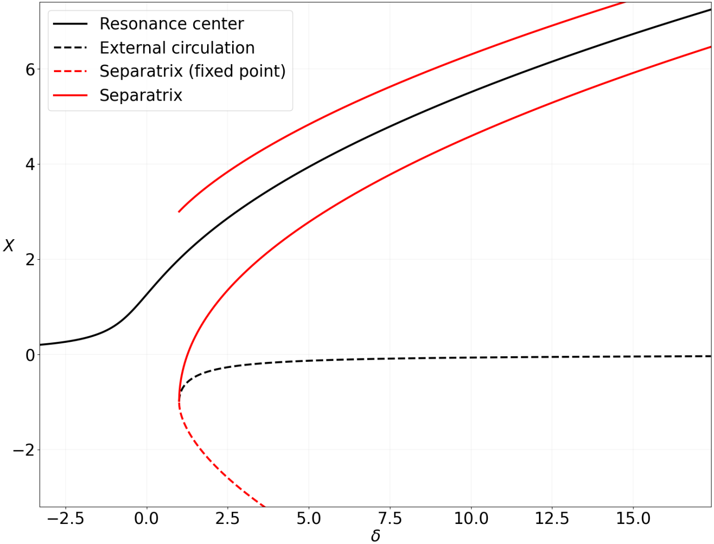
$\delta = 4$



Subspace $Y = 0$


Resonance center
External circulation
Separatrix (fixed point)
Separatrix
X
6
4
2
0
−2
−2.5
0.0
2.5
5.0
7.5
10.0
12.5
15.0
δ

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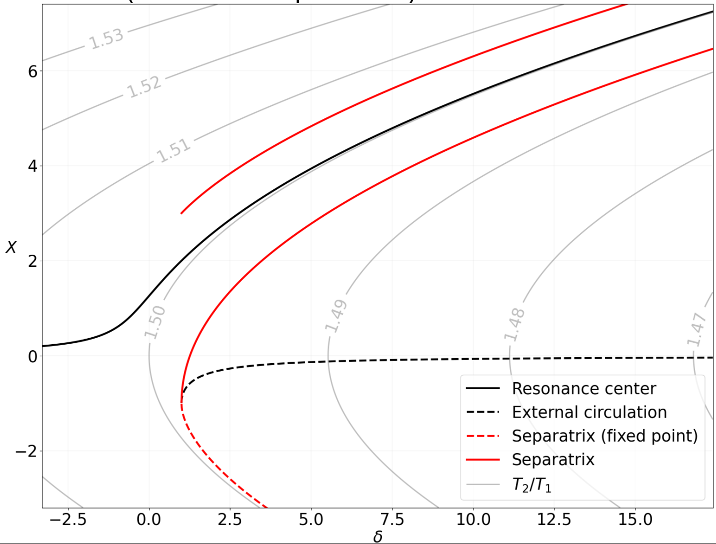
Subspace $Y = 0$ (masses of Kepler-1972)



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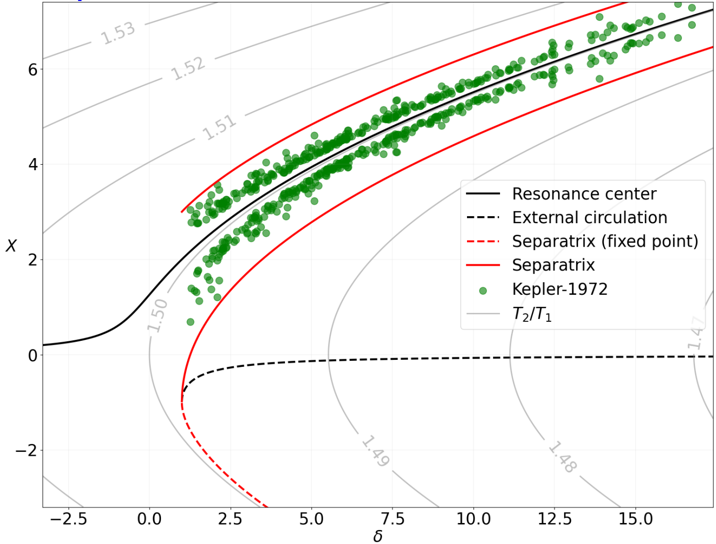
Kepler-1972 posteriors



Overly simplified model. What about a real system?

Many approximations, including:

- Perturbations of the pair by other planets when $N > 2$ are disregarded.
- Model limited to first-order in eccentricity
- Perturbative part simply evaluated at $a_1/a_2 = (p/(p+1))^{2/3}$
- Averaging over the fast angles of the Hamiltonian
- Contributions in $\mathcal{O}(m_j/m_0)^2$ disregarded

I am creating a software that

- Automatically studies a given chain $p_1 : p_2 : \cdots : p_n$
- Can manipulate both the averaged and complete Hamiltonian
- Can find fixed points
- Can find *libration centers*

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Aptidal (name might change, IDK)

```
The resonance chain to be studied is 3:4:6
The system currently has 6 degrees of freedom.

Aptidal will perform a canonical transformation (lbd, -vrp; Lbd, D) -> (phi, sig; Phi, D)
where (lbd, -vrp; Lbd, D) are the traditional Poincaré variables.
lbd (lambda) is the mean longitude, vrp (varpi) is the longitude of the ascending node, Lambda (resp. Phi)
is the action conjugated to lambda (resp. phi) and D (the AMD) is conjugated to both sigma and -varpi.

The transformation reads

 phi_1 = lbd_1 - 2 lbd_2 + lbd_3
 phi_2 = lbd_2 - lbd_3
 phi_3 = - 2 lbd_2 + 3 lbd_3
 sig_j = - 2 lbd_2 + 3 lbd_3 - vrp_j,  1 <= j <= 3

It is canonical if the actions are transformed according to (the D_j are unchanged)

 Phi_1 = Lbd_1
 Phi_2 = 4 Lbd_1 + 3 Lbd_2 + 2 Lbd_3
 Phi_3 = Lbd_1 + Lbd_2 + Lbd_3 - D_1 - D_2 - D_3
```

```
After averaging over the fast angle, the 4 remaining degrees of freedom are

 (phi_1; Phi_1), (sig_1; D_1), (sig_2; D_2), (sig_3; D_3)
```

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Aptidal (name might change, IDK)

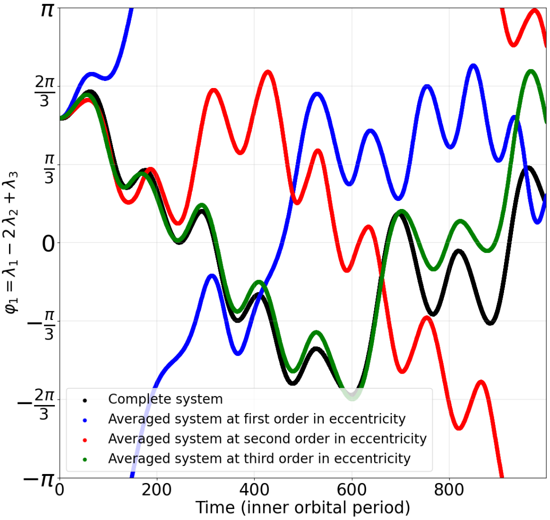
```
H = - sum_{1 <= j <= 3} beta_j**3 * mu_j**2 / (2*Lbd_j**2)

+ G*m_1*m_2/a_2 * (
    - 2.1973798573 * sqrt(2*D_1/Lbd_1)**2
    - 2.1973798573 * sqrt(2*D_2/Lbd_2)**2
    + 4.0222128577 * sqrt(2*D_1/Lbd_1)    * sqrt(2*D_2/Lbd_2)    * cos(vrp_1 - vrp_2)
    + 2.7760167116 * sqrt(2*D_1/Lbd_1)                           * cos(3 lbd_1 - 4 lbd_2 + vrp_1)
    - 3.2201137810 * sqrt(2*D_2/Lbd_2)                           * cos(3 lbd_1 - 4 lbd_2 + vrp_2)
    - 10.472815452 * sqrt(2*D_1/Lbd_1)**2                        * cos(6 lbd_1 - 8 lbd_2 + 2 vrp_1)
    + 24.574405534 * sqrt(2*D_1/Lbd_1)    * sqrt(2*D_2/Lbd_2)    * cos(6 lbd_1 - 8 lbd_2 + vrp_1 + vrp_2)
    - 14.385904679 * sqrt(2*D_2/Lbd_2)**2                        * cos(6 lbd_1 - 8 lbd_2 + 2 vrp_2)
)

+ G*m_1*m_3/a_3 * (
    - 0.3693855360 * sqrt(2*D_1/Lbd_1)**2
    - 0.3693855360 * sqrt(2*D_3/Lbd_3)**2
    + 0.5435235753 * sqrt(2*D_1/Lbd_1)    * sqrt(2*D_3/Lbd_3)    * cos(vrp_1 - vrp_3)
    + 1.1556528318 * sqrt(2*D_1/Lbd_1)                           * cos(lbd_1 - 2 lbd_3 + vrp_1)
    - 0.3896164810 * sqrt(2*D_3/Lbd_3)                           * cos(lbd_1 - 2 lbd_3 + vrp_3)
    - 1.6079268247 * sqrt(2*D_1/Lbd_1)**2                        * cos(2 lbd_1 - 4 lbd_3 + 2 vrp_1)
    + 4.7599824848 * sqrt(2*D_1/Lbd_1)    * sqrt(2*D_3/Lbd_3)    * cos(2 lbd_1 - 4 lbd_3 + vrp_1 + vrp_3)
    - 3.4807825350 * sqrt(2*D_3/Lbd_3)**2                        * cos(2 lbd_1 - 4 lbd_3 + 2 vrp_3)
)

+ G*m_2*m_3/a_3 * (
    - 1.0937567733 * sqrt(2*D_2/Lbd_2)**2
    - 1.0937567733 * sqrt(2*D_3/Lbd_3)**2
    + 1.8878074721 * sqrt(2*D_2/Lbd_2)    * sqrt(2*D_3/Lbd_3)    * cos(vrp_2 - vrp_3)
    + 1.9595502701 * sqrt(2*D_2/Lbd_2)                           * cos(2 lbd_2 - 3 lbd_3 + vrp_2)
    - 2.4205566721 * sqrt(2*D_3/Lbd_3)                           * cos(2 lbd_2 - 3 lbd_3 + vrp_3)
    - 5.0371064812 * sqrt(2*D_2/Lbd_2)**2                        * cos(4 lbd_2 - 6 lbd_3 + 2 vrp_2)
    + 12.655477566 * sqrt(2*D_2/Lbd_2)    * sqrt(2*D_3/Lbd_3)    * cos(4 lbd_2 - 6 lbd_3 + vrp_2 + vrp_3)
    - 7.9162336115 * sqrt(2*D_3/Lbd_3)**2                        * cos(4 lbd_2 - 6 lbd_3 + 2 vrp_3)
)
```

Can integrate the model and the real system automatically



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Fixed points vs libration center ? Chain $3:4:6:8$ Fixed points

- $\phi_1 = \lambda_1 - 2\lambda_2 + \lambda_3 =$ Cte
- $\phi_2 = \lambda_2 - 3\lambda_3 + 2\lambda_4 =$ Cte
- $\phi_3 = \lambda_3 - \lambda_4$
- $\phi_4 = -3\lambda_3 + 4\lambda_4$
- $\sigma_j = -3\lambda_3 + 4\lambda_4 - \varpi_j =$ Cte

Libration center

- $\phi_1 = \lambda_1 - 2\lambda_2 + \lambda_3 =$ is periodic of period ν_3
- $\phi_2 = \lambda_2 - 3\lambda_3 + 2\lambda_4 =$ is periodic of period ν_3
- $\phi_3 = \lambda_3 - \lambda_4 = \nu_3 t +$ periodic of period ν_3
- $\phi_4 = -3\lambda_3 + 4\lambda_4 = \nu_4 t +$ periodic of period ν_3
- $\sigma_j = -3\lambda_3 + 4\lambda_4 - \varpi_j =$ is periodic of period ν_3

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Can find fixed points of the model

```
Starting the search for a fixed point.

Iteration n° 0 : (lbd_1, lbd_2, lbd_3) = (0.9999991553361, 0.0000000000000, 0.0000000000000) x pi
                 (vrp_1, vrp_2, vrp_3) = (0.6816892691523, 0.0000000000000, 0.9999991553361) x pi
                 (a_1, a_2, a_3)       = (0.9999646060000, 1.2157125170000, 1.6045573520000)
                 (e_1, e_2, e_3)       = (0.0039254030000, 0.0035638750000, 0.0014761090000)

Iteration n° 1 : (lbd_1, lbd_2, lbd_3) = (1.0000048947343, 0.0000000000000, 0.0000000000000) x pi
                 (vrp_1, vrp_2, vrp_3) = (0.9999018601172, 0.0000772588654, 0.9998723307605) x pi
                 (a_1, a_2, a_3)       = (0.9999528690489, 1.2157312263377, 1.6045437220038)
                 (e_1, e_2, e_3)       = (0.0039117085973, 0.0035517536071, 0.0014708166399)
```

```
Iteration n° 6 : (lbd_1, lbd_2, lbd_3) = (0.9999999999989, 0.0000000000000, 0.0000000000000) x pi
                 (vrp_1, vrp_2, vrp_3) = (-0.9999999999973, 0.0000000000021, 0.9999999999999) x pi
                 (a_1, a_2, a_3)       = (0.9999646061216, 1.2157125167514, 1.6045573522558)
                 (e_1, e_2, e_3)       = (0.0039254046462, 0.0035638770699, 0.0014761094377)
```

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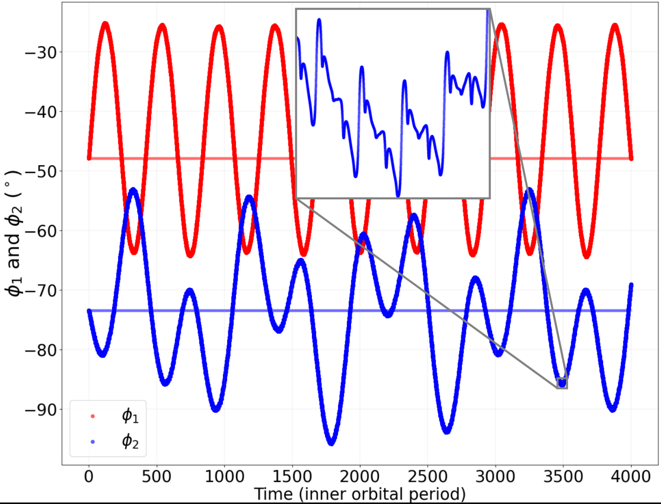
Finding a fixed point

- By successive average of a variable $f(t)$ of the Hamiltonian



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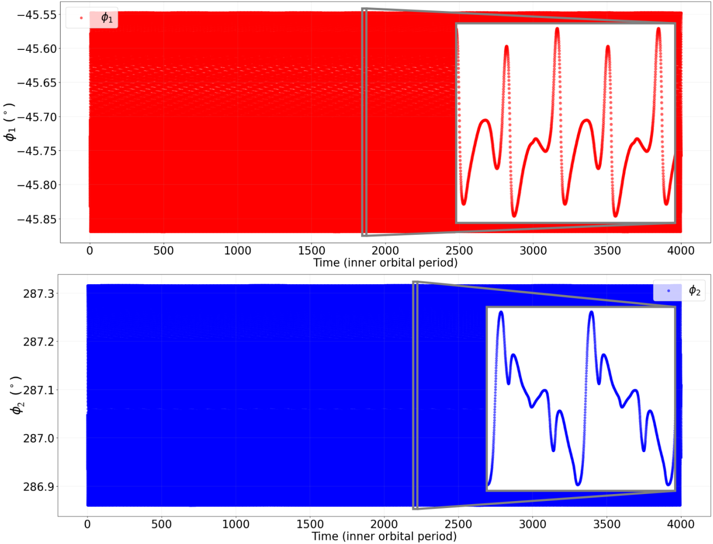
Converging to a libration center

- The system is integrated in the complete system using a fixed point of the model as initial condition
- $f(t) = \sum_{\mathbf{k} \in \mathbb{Z}^8} f_{\mathbf{k}} e^{i\mathbf{k}\cdot\boldsymbol{\omega} t}$
- $\boldsymbol{\omega} = (\nu_1, \nu_2, \cdots, \nu_8)$ vector of fundamental frequencies.
- $f_{\mathbf{k}}$ is set to 0 except if $\mathbf{k} = (0, 0, k, 0, 0, 0, 0, 0)$ for $k \in \mathbb{Z}$.
- Done on all variables of the Hamiltonian
- Defines a new initial condition from which the process is reiterated.

Computing $f_{\mathbf{k}}$: NAFF algorithm

- The system is integrated from $t = -T$ to $t = T$
- $\nu = \mathbf{k} \cdot \boldsymbol{\omega}$
- $f_{\mathbf{k}} = \dfrac{1}{2T} \displaystyle\int_{-T}^{T} f(t) e^{-i\nu t} H_p(t) dt$
- $\nu = k\nu_3 = (0, 0, k, 0, 0, 0, 0, 0) \cdot (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7, \nu_8)$
- ν_3 obtained from a linear fit of $\phi_3(t) = \nu_3 t + \phi_3(0)$

Libration center



Periodic orbit finding

- Variables of the Hamiltonian periodic with frequency $\nu_3 = 0.79780724$
- In a frame rotation with the pericenters at frequency $\nu_4 = 0.09929880$
- $\nu_3/\nu_4 = 8.0344$
- If $\nu_3/\nu_4 \in \mathbb{Q}$, then the motion is periodic
- A whole family of libration centers parameterized by the total angular momentum
- Periodic orbits can be found by varying the angular momentum until $\nu_3/\nu_4 \in \mathbb{Q}$