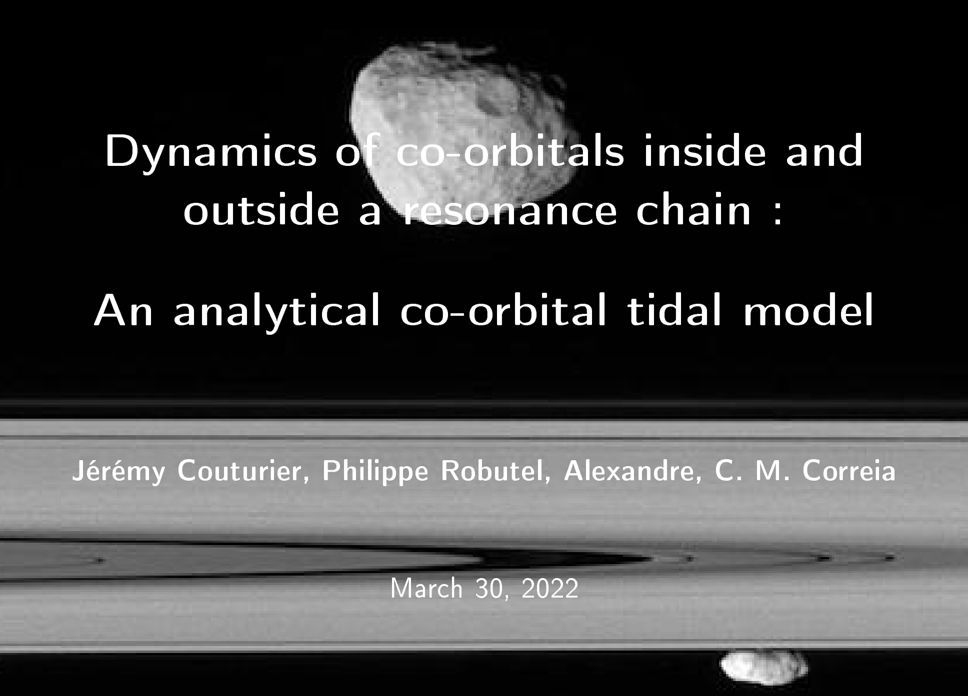

Dynamics of co-orbitals inside and outside a resonance chain :

An analytical co-orbital tidal model

Jérémy Couturier, Philippe Robutel, Alexandre, C. M. Correia

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Some example of co-orbital bodies in the solar system

- Trojans of Jupiter (first discovery by *Wolf, 1906*).
- Some satellites of Saturn
- The famous triplet Saturn-Janus-Epimetheus, only example with comparable masses.

But no co-orbital planets.

Co-orbital exoplanets are predicted by the formation model
Two scenarii of formation (*Laughlin and Chambers, 2002*)

- Accretion in situ at the Lagrangian point of a primary giant planet
  $\rightarrow$ maximum 0.6 Earth mass according to *Beaugé et al. (2007)*.
  $\rightarrow$ 5 to 15 Earth mass according to *Lyra et al. (2009)*.
- Capture at the L_{4-5} point of an already existing planet
  $\rightarrow$ a high diversity of mass ratio (*Cresswell and Nelson, 2008*)

But their stability after the formation is not guaranteed
$\rightarrow$ *Pierens and Raymond 2014, Leleu et al. 2019*

Assuming the existence, the detection is challenging

- Mutual inclination $\rightarrow$ only one body transits.
- High mass ratio $\rightarrow$ only one body perturbs the radial velocity of the star.
- Large orbital period $\rightarrow$ only few transits per unit time & necessity of an almost zero mutual inclination.
- Low orbital period $\rightarrow$ strong tidal interaction with the host star

Identical co-orbital (same mass, radius, tidal parameters) are always destroyed by tides (*Rodriguez et al., 2013, using numerical simulations)*

→ The libration amplitude increases until close encounters between the planets destroys the configuration.

No analytical work has been undertaken so far.

Do tides always disrupt the system ? → Yes

On what timescale ? → Analytical expression depending on parameters

Hamiltonian of the conservative problem

$$\mathcal{H} = \mathcal{H}_K(\Lambda_1, \Lambda_2) + H_P(\Lambda_1, \Lambda_2, \lambda_1, \lambda_2, x_1, x_2, \tilde{x}_1, \tilde{x}_2), \tag{1}$$

$$\mathcal{H}_K(\Lambda_1, \Lambda_2) = -\sum_{j=1}^{2} \frac{\beta_j^3 \mu_j^2}{2\Lambda_j^2}. \tag{2}$$

The semi-major axes stay close to a quantity denoted by $\bar{a}$.

$\rightarrow$ Expansion in the neighbourhood of

$$\mathbf{\Lambda}^\star = (\Lambda_1^\star, \Lambda_2^\star)\,, \quad \text{with} \quad \Lambda_j^\star = m_j\sqrt{\mu_0 \bar{a}} \approx \Lambda_j. \tag{3}$$

Averaging over the fast orbital frequency

$$(\Lambda_1, \Lambda_2, \lambda_1, \lambda_2) \longmapsto (Z, Z_2, \phi, \phi_2) = (\Lambda_1 - \Lambda_1^\star,\ \Lambda_1 + \Lambda_2 - \Lambda_1^\star - \Lambda_2^\star,\ \lambda_1 - \lambda_2,\ \lambda_2).$$

$\rightarrow$ **Average over ϕ_2.**

Expansion of $\mathcal{H}_P$

$$\mathcal{H}_P = \sum_{n\geq 0} \mathcal{H}_{2n} \quad \text{where} \quad \mathcal{H}_{2n} = \sum_{|\mathbf{p}|=2n} \Psi_{\mathbf{p}}\left(\xi\right) X_1^{p_1} X_2^{p_2} \bar{X}_1^{\bar{p}_1} \bar{X}_2^{\bar{p}_2} . \tag{4}$$

$$\begin{aligned} & p_1 + p_2 = \bar{p}_1 + \bar{p}_2, \qquad \text{(D'alembert rule)},\\ & \mathcal{H}_0 = \frac{m}{m_0}\left(\cos\xi - \left(2 - 2\cos\xi\right)^{-1/2}\right), \qquad \boxed{\xi = \lambda_1 - \lambda_2}\\ & \mathcal{H}_2 = \frac{1}{2}\frac{m}{m_0}\left\{A_h\left(\xi\right)\left(X_1\bar{X}_1 + X_2\bar{X}_2\right) + B_h\left(\xi\right)X_1\bar{X}_2 + \bar{B}_h\left(\xi\right)\bar{X}_1 X_2\right\}. \end{aligned} \tag{5}$$

X_j close to the eccentricity vector

$$X_j \approx e_j \exp(i\varpi_j). \tag{6}$$

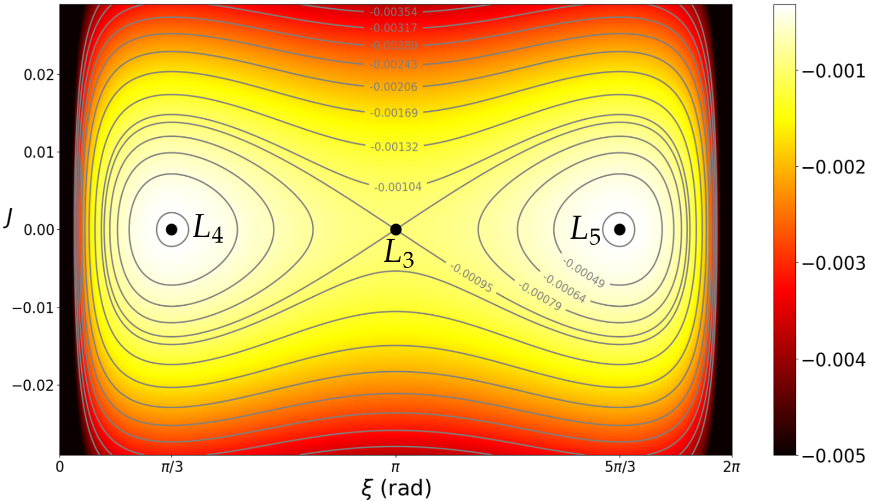
Invariance by rotation $\rightarrow$ D'alembert rule $\rightarrow$ $\mathcal{H}_{2n+1} = 0$
$\rightarrow$ $X_1 = X_2 = 0$ is a stable manifold of the space phase.

What about the dynamics at zero eccentricity ?

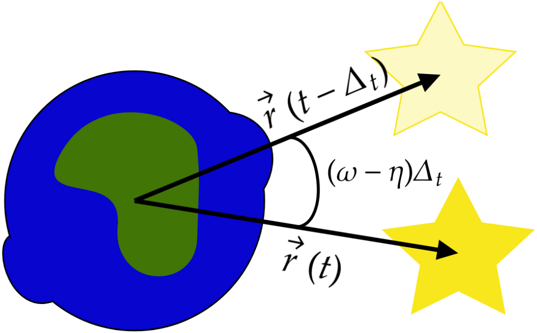
We study the one-degree-of-freedom Hamiltonian $\mathcal{H}_K + \mathcal{H}_0$

Space phase in the circular case

L_4 and L_5 are energy maximizers $\rightarrow$ L_4 and L_5 are tidally unstable.



The Mignard model, Mignard, 1979.



$$V(\mathbf{r}) = -\kappa_2^{(j)}\frac{\mathcal{G}m_i}{R_j}\left(\frac{R_j}{r}\right)^3\left(\frac{R_j}{r_i^\star}\right)^3 P_2\left(\cos S\right), \qquad \mathbf{r_i}^\star = \mathbf{r_i}\left(t-\Delta t_j\right).$$

Quality factor

$$Q_j^{-1}(\eta) = \sin(\eta\Delta t_j(\eta)) \approx \eta\Delta t_j(\eta). \tag{7}$$

Tidal (pseudo) Hamiltonian

$$\mathcal{H}_t = \mathcal{H}_t^1 + \mathcal{H}_t^2, \tag{8}$$

$$\mathcal{H}_t^j = -\kappa_2^{(j)} \mathcal{G} m_0^2 \frac{R_j^5}{r_j^3 r_j^{\star 3}} P_2\left(\cos S\right), \qquad S = \lambda_j - \lambda_j^{\star} - \left(\theta_j - \theta_j^{\star}\right). \tag{9}$$

We define the dissipation rate [1]

$$\text{dissipation rate of planet } j: \; q_j/Q_j = \kappa_2^{(j)} \frac{R_j^5}{\bar{a}^5} \eta \Delta t_j. \tag{10}$$

We obtain an equation of the form [1]

$$\dot{\mathcal{X}} = F\left(\mathcal{X}\right), \qquad \mathcal{X} = {}^t\left(\omega_1, \omega_2, J, J_2, \xi, X_1, X_2\right). \tag{11}$$

[1] *Couturier, Robutel & Correia, 2021*

Linearization of the differential system

in the vicinity of its equilibria $\rightarrow$ Small perturbation of L_4 & L_5.

Linear system

$$\dot{\mathcal{X}} = \left(\mathcal{Q}_0 + \mathcal{Q}_1\right)\mathcal{X}. \tag{12}$$

$\mathcal{Q}_0$ conservative part, $\mathcal{Q}_1$ dissipative part.

$$\mathcal{Q}_{0 \text{ or } 1} = \begin{pmatrix} * & 0_{5,2} \\ 0_{2,5} & * \end{pmatrix} \tag{13}$$

$\rightarrow$ Even in the dissipative case, the dynamics on (X_1, X_2) is uncoupled from the rest in the vicinity of the equilibrium.

In the absence of tides

Eigenvalues of $\mathcal{Q}_0$ (*Robutel & Pousse, 2013*)

$$\{0,0,0,i\nu,-i\nu,ig_1,ig_2\}\,,\qquad \nu=\sqrt{\frac{27\varepsilon}{4}},\qquad g_1=\frac{27\varepsilon}{8},\qquad g_2=0. \tag{14}$$

All the eigenvalues are pure imaginary $\rightarrow$ Quasi-periodic motion

With tidal dissipation

Eigenvalues of $\mathcal{Q}_0 + \mathcal{Q}_1$ (*Couturier, Robutel & Correia, 2021*)

$$\left\{\lambda_1, \lambda_2, 0, \lambdabar, \bar{\lambdabar}, \lambda_{\mathsf{AL}}, \lambda_{\mathsf{L}}\right\}, \tag{15}$$

with (real parts are boxed)

$$\lambda_j = \boxed{-3\alpha_j^{-1}\frac{q_j}{Q_j}\varrho_j^{-2}\frac{m_0}{m_j} + 9\varepsilon^{-1}\frac{q_j}{Q_j}} < 0,$$

$$\lambdabar = \boxed{\frac{9}{2}\varepsilon^{-1}\left(\frac{m_1}{m_2}\frac{q_2}{Q_2} + \frac{m_2}{m_1}\frac{q_1}{Q_1}\right)} + i\nu\left[1 + 13\varepsilon^{-1}\left(\frac{m_1}{m_2}q_2 + \frac{m_2}{m_1}q_1\right)\right],$$

$$\lambda_{\mathsf{AL}} = \boxed{-\frac{21}{2}\varepsilon^{-1}\left(\frac{m_1}{m_2}\frac{q_2}{Q_2} + \frac{m_2}{m_1}\frac{q_1}{Q_1}\right)} + ig_1\left[1 + \frac{20}{9}\varepsilon^{-2}\left(\frac{m_1}{m_2}q_2 + \frac{m_2}{m_1}q_1\right)\right],$$

$$\lambda_{\mathsf{L}} = \boxed{-\frac{21}{2}\varepsilon^{-1}\left(\frac{q_1}{Q_1} + \frac{q_2}{Q_2}\right)} + \frac{15}{2}i\varepsilon^{-1}\left(q_1 + q_2\right).$$

Characteristic timescales of evolution of the co-orbital angle[1]

$$\tau_{\text{lib}} = \frac{1}{9\pi}\frac{\varepsilon}{q_1/Q_1 + q_2/Q_2}\frac{x\,(1+y)}{1+yx^2}T \propto \frac{x\,(1+y)}{1+yx^2}\bar{a}^{6.5}. \tag{16}$$

with

$$x = \frac{m_1}{m_2} \text{ and } y = \frac{\text{dissipation rate}_2}{\text{dissipation rate}_1} = \frac{q_2 Q_1}{q_1 Q_2} \tag{17}$$

Time to horse-shoe shaped orbits

$$\tau_{\text{hs}} = \ln\left(\frac{60^\circ}{\Delta\xi}\right)\tau_{\text{lib}} \tag{18}$$

where $\Delta\xi$ is the initial angular distance to L_{4-5}

[1] *Couturier, Robutel & Correia, 2021*

Co-orbital lifetime

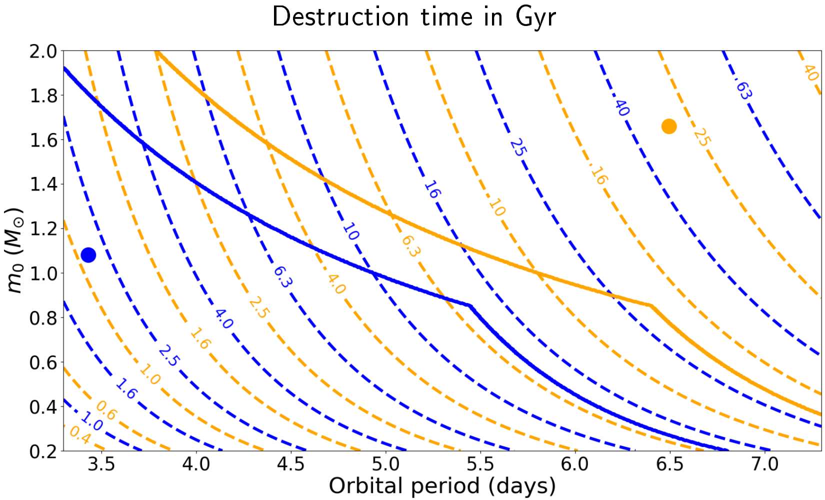
$$\tau_{\text{dest}} \approx \tau_{\text{hs}} = \frac{\varepsilon}{9\pi\Omega}\frac{x\,(1+y)}{1+yx^2}\ln\left(\frac{60^\circ}{\Delta\xi}\right)T, \tag{19}$$

For two co-orbital Earth-like planets initially 0.1° away from L_4 :

$$\tau_{\text{hs}} = 3.771\ \text{Gyr}\ \left(\frac{\bar{a}}{0.04\ \text{AU}}\right)^{6.5}\left(\frac{m_0}{m_\odot}\right)^{-3/2}, \tag{20}$$

Co-orbital pair	τ_{hs} (Gyr)	Co-orbital pair	τ_{hs} (Gyr)
Earth & Earth	3.771	Earth & Moon	44.09
Earth & Mars	5.480	Earth & Jupiter	3.722
Earth & Io	2.086	Moon & Moon	50.76
Moon & Mars	28.16	Moon & Jupiter	50.63
Moon & Io	4.374	Mars & Mars	5.761
Mars & Jupiter	5.747	Mars & Io	2.248
Jupiter & Jupiter	0.7201	Jupiter & Io	2.072
Io & Io	2.072		

A tool to help with the detection of co-orbital exoplanets



Solid line $=$ $\min$ (main sequence duration, universe age)

Orange dot $\rightarrow$ gas giant HD 102956 b

Blue dot $\rightarrow$ rocky planet HD 158259 c

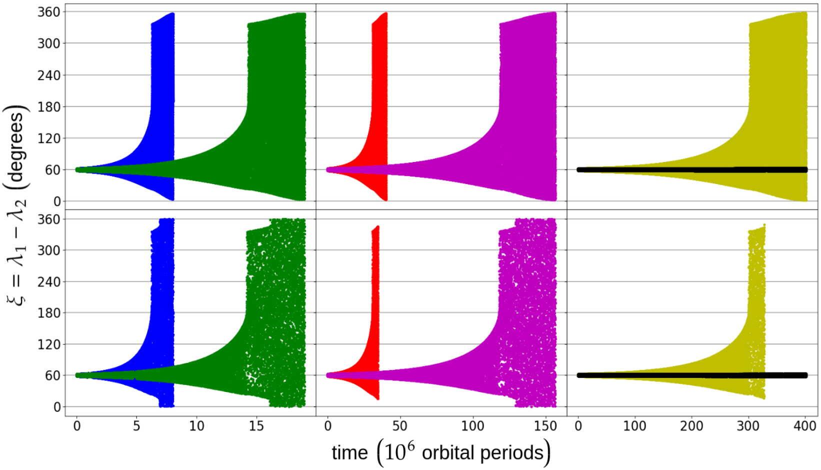
6 systems are numerically integrated

#	color	x	y	τ_{lib}
1	blue	10	100	1 845 021
2	green	$1/500$	100	4 198 710
3	red	100	$1/50$	9 161 859
4	purple	100	$1/200$	34 893 952
5	yellow	$1/10$	100	89 304 263
6	black	100	10^{-5}	1 607 642 323

Table: Characteristic timescales of increase of the libration amplitude of ξ, expressed in number of orbital periods.

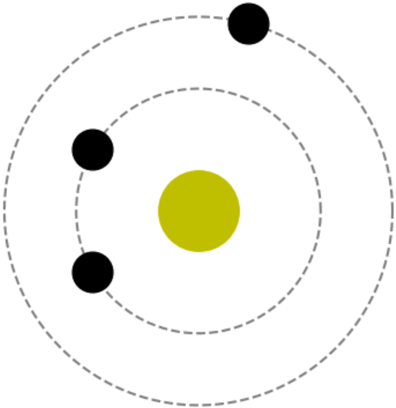
- Top panel → Our model
- Bottom panel → Full n-body problem

Libration amplitude → unbounded



Less than 1% error on time to horseshoe-shaped orbits.

We introduce a third planet in the planetary system



First order resonance between the co-orbital pair and the third planet.

The co-orbitals complete $p + 1$ orbits while the third planet completes p orbits.

The resonance chain $p:p:p+1$

Angles of the form

$$-p\lambda_1 + (p+1)\,\lambda_3 \quad \text{or} \quad -p\lambda_2 + (p+1)\,\lambda_3$$

are secularly evolving and cannot be averaged. The averaged Hamiltonian contains terms of the form

$$e_j\,\cos\left(-p\lambda_j + (p+1)\,\lambda_3 - \varpi_j\right) \text{ and } e_3\,\cos\left(-p\lambda_j + (p+1)\,\lambda_3 - \varpi_3\right)$$

$+$ more terms at second order in eccentricity

4 degrees of freedom associated with the angles

$$\begin{aligned}
\xi &= \lambda_1 - \lambda_2 \\
\sigma_1 &= -p\lambda_2 + (p+1)\,\lambda_3 - \varpi_1 \\
\sigma_2 &= -p\lambda_2 + (p+1)\,\lambda_3 - \varpi_2 \\
\sigma_3 &= -p\lambda_2 + (p+1)\,\lambda_3 - \varpi_3
\end{aligned}$$

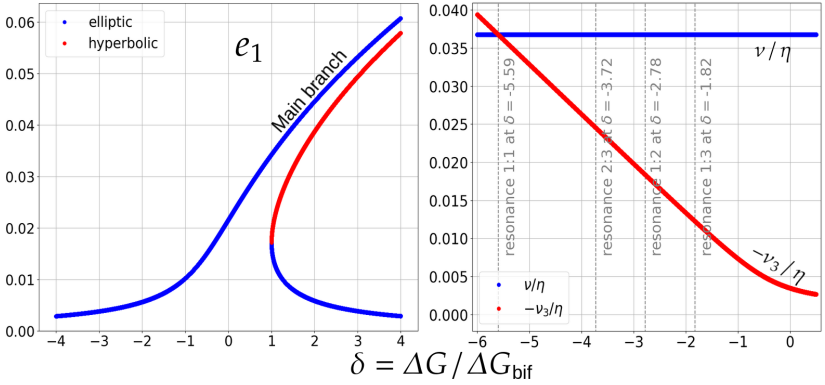


Figure: Left : Value of e_1 at the analytical equilibria of the resonance chain $1:1:2$. Right: Libration frequency of ξ (blue) and ϖ_j (red).

The problem depends on a parameter δ related to a_1/a_3, that is, to n_1/n_3

- $\delta > 0 \rightarrow \frac{n_1}{n_3} \approx \frac{p+1}{p} \rightarrow$ system close to the Keplerian resonance
- $\delta < 0 \rightarrow \frac{n_1}{n_3} > \frac{p+1}{p} \rightarrow$ system far from the Keplerian resonance

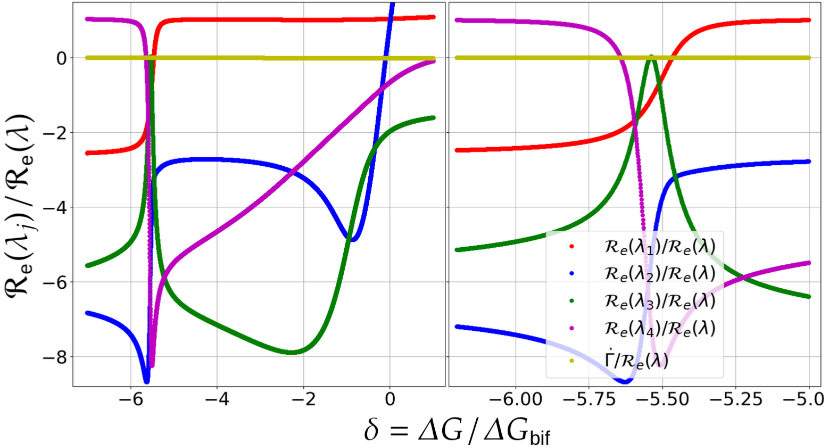


Figure: Real parts of the linearized differential system in the vicinity of the equilibria, with tidal dissipation.

At the $1:1$ secular resonance between the libration frequency of ξ and the precession frquency of the pericentres $\rightarrow$ Linear stability

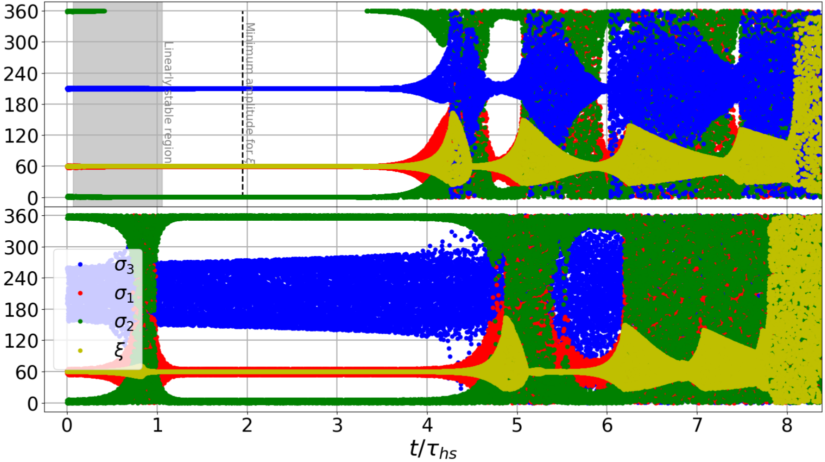


Figure: Evolution of the angles as a function of time

Conclusion

Co-orbital exoplanets are always unstable under tides.

But they live long if :

- They orbit far away from the star → challenging detection
- The mass repartition is very inequal → challenging detection
- They are within the resonance chain $p : p : p + 1$

We have a satisfactory explanation as to why no co-orbital planet has been detected so far, even though formation models predict their existence.

Much more details in the associated papers :

- An analytical model for tidal evolution in co-orbital systems *10.1007/s10569-021-10032-w CM&DA*
- Dynamics of co-orbital exoplanets in a first order resonance chain with tidal dissipation (under review at A&A)

Thank you for your attention

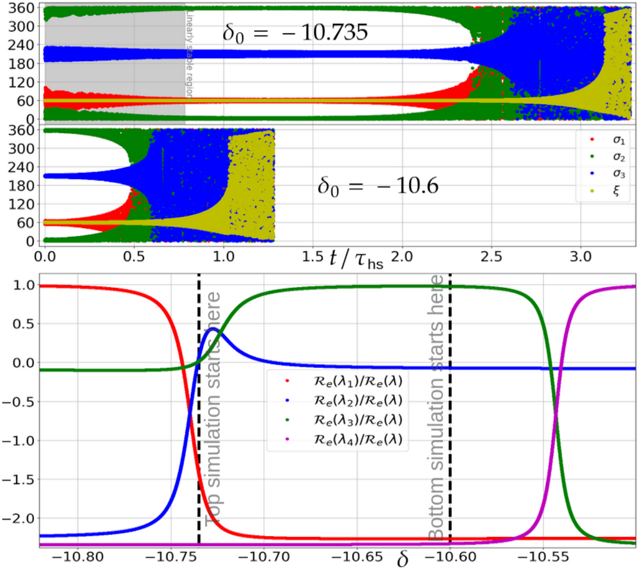


Figure: $m_3 = (m_1 + m_2)\,/32$

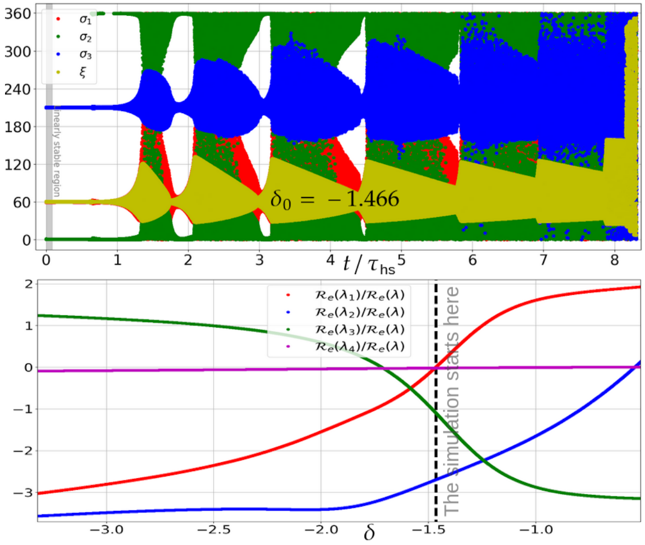


Figure: $m_3 = 8\,(m_1 + m_2)$

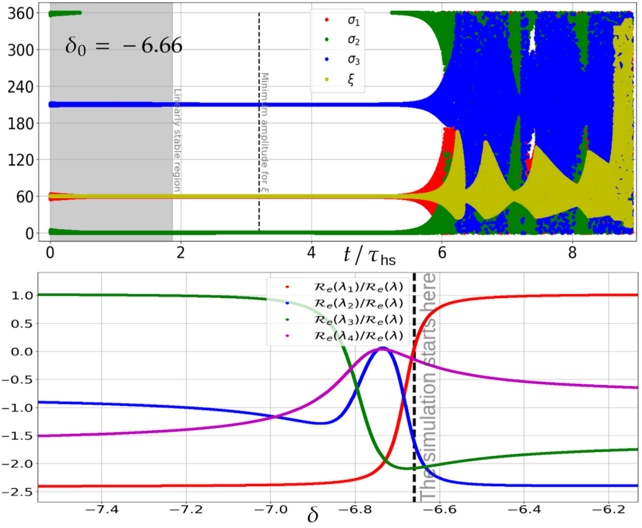


Figure: $m_3 = 0.29\,(m_1 + m_2)$

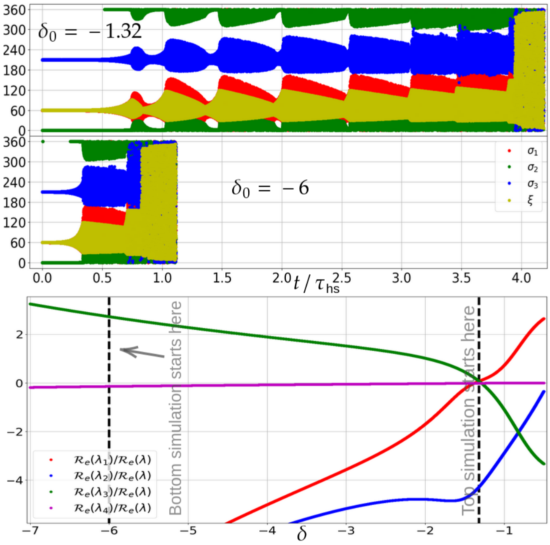


Figure: $m_3 = 19\,(m_1 + m_2)$

Two different sets of equations

- Top plot → the secular equations derived in this work.
- Bottom plot → A direct n-body model in cartesian coordinates.

$$\frac{d^2\boldsymbol{r}_1}{dt^2} = -\frac{\mu_1}{r_1^3}\boldsymbol{r}_1 + \mathcal{G}m_2\left(\frac{\boldsymbol{r}_2-\boldsymbol{r}_1}{|\boldsymbol{r}_2-\boldsymbol{r}_1|^3} - \frac{\boldsymbol{r}_2}{r_2^3}\right) + \frac{\boldsymbol{f}_1}{\beta_1} + \frac{\boldsymbol{f}_2}{m_0} \ ,$$

$$\frac{d^2\boldsymbol{r}_2}{dt^2} = -\frac{\mu_2}{r_2^3}\boldsymbol{r}_2 + \mathcal{G}m_1\left(\frac{\boldsymbol{r}_1-\boldsymbol{r}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3} - \frac{\boldsymbol{r}_1}{r_1^3}\right) + \frac{\boldsymbol{f}_2}{\beta_2} + \frac{\boldsymbol{f}_1}{m_0} \ ,$$

$$\frac{d^2\theta_i}{dt^2} = -\frac{(\boldsymbol{r}_i\times\boldsymbol{f}_i)\cdot\boldsymbol{k}}{C_i} = -3\frac{\kappa_{2,i}\mathcal{G}m_0^2R_i^3}{\alpha_i m_i r_i^8}\Delta t_i\left[\frac{d\theta_i}{dt}\,r_i^2 - \left(\boldsymbol{r}_i\times\frac{d\boldsymbol{r}_i}{dt}\right)\cdot\boldsymbol{k}\right] \ ,$$

$$\boldsymbol{f}_i = -3\frac{\kappa_{2,i}\mathcal{G}m_0^2R_i^5}{r_i^8}\boldsymbol{r}_i$$

$$-3\frac{\kappa_{2,i}\mathcal{G}m_0^2R_i^5}{r_i^{10}}\Delta t_i\left[2\left(\boldsymbol{r}_i\cdot\frac{d\boldsymbol{r}_i}{dt}\right)\boldsymbol{r}_i + r_i^2\left(\frac{d\theta_i}{dt}\,\boldsymbol{r}_i\times\boldsymbol{k} + \frac{d\boldsymbol{r}_i}{dt}\right)\right] \ .$$

The secular equations

$$\dot{\vartheta}_j = -3\alpha_j^{-1}\frac{m_0}{m_j}\text{ϙ}_j^{-2}\frac{q_j}{Q_j}\mathcal{R}_j^{-12}\left\{\vartheta_j + 3\left(1-\mathcal{R}_j\right) + h_2^j\mathcal{R}_j^{-1}X_j\bar{X}_j\right.$$
$$\left. + h_4^j\mathcal{R}_j^{-2}X_j^2\bar{X}_j^2\right\},$$
$$\dot{J} = -\frac{\partial\left(\mathcal{H}_0+\mathcal{H}_2+\mathcal{H}_4\right)}{\partial\xi} + \left(1-\delta\right)\dot{J}_2^1 - \delta\dot{J}_2^2,$$
$$\dot{J}_2 = \dot{J}_2^1 + \dot{J}_2^2,$$
$$\dot{\xi} = \frac{\partial\mathcal{H}_K}{\partial J} + 6q_1\frac{m_0}{m_1}\mathcal{R}_1^{-13}V_2\left(\mathcal{R}_1^{-1}X_1\bar{X}_1\right) - 6q_2\frac{m_0}{m_2}\mathcal{R}_2^{-13}V_2\left(\mathcal{R}_2^{-1}X_2\bar{X}_2\right),$$
$$\dot{X}_j = -2i\frac{m}{m_j}\frac{\partial\left(\mathcal{H}_2+\mathcal{H}_4\right)}{\partial\bar{X}_j}$$
$$-3\frac{q_j}{Q_j}\frac{m_0}{m_j}\mathcal{R}_j^{-13}X_j\left\{p_2^j - \frac{5}{2}iQ_j + \frac{X_j\bar{X}_j}{\mathcal{R}_j}\left(p_4^j - \frac{65}{4}iQ_j\right)\right\},$$