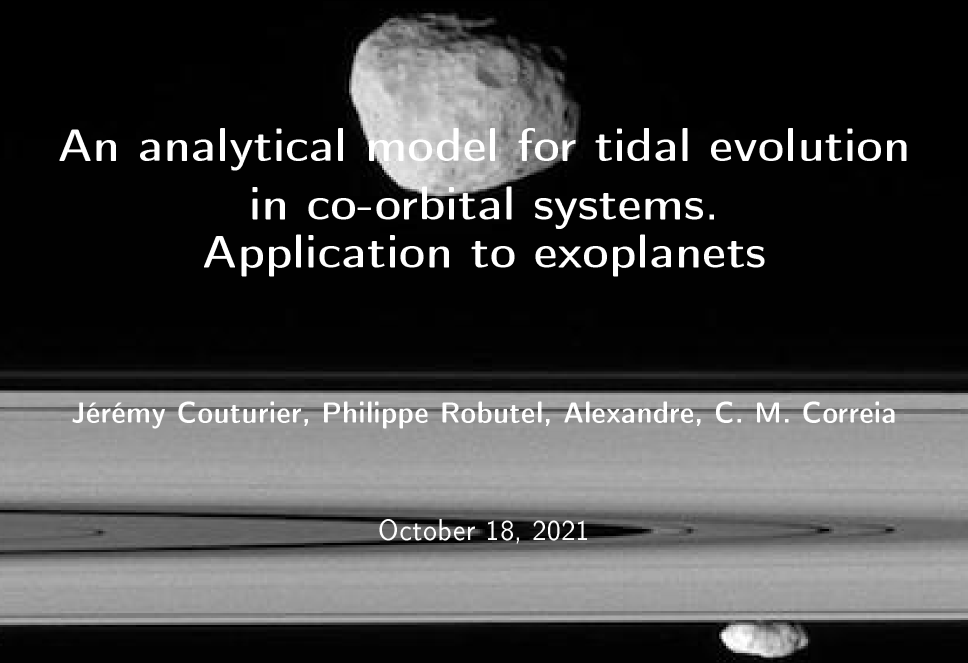

An analytical model for tidal evolution in co-orbital systems. Application to exoplanets

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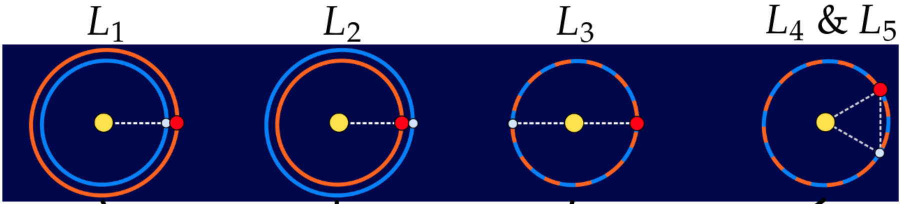
Euler & Lagrange fixed points



Unstable whatever the planetary masses

Stable in the conservative case provided the Gascheau condition is fulfilled : $\frac{m_1 + m_2}{m_0} < \frac{1}{27}$

Some example of co-orbital bodies in the solar system

- Trojans of Jupiter (first discovery by *Wolf, 1906*).
- Some satellites of Saturn
- The famous triplet Saturn-Janus-Epimetheus, only example with comparable masses.

But no co-orbital planets.

Co-orbital exoplanets are predicted by the formation model
Two scenarii of formation (*Laughlin and Chambers, 2002*)

- Accretion in situ at the Lagrangian point of a primary giant planet
  $\rightarrow$ maximum 0.6 Earth mass according to *Beaugé et al. (2007)*.
  $\rightarrow$ 5 to 15 Earth mass according to *Lyra et al. (2009)*.
- Planet-planet gravitational scattering
  $\rightarrow$ a high diversity of mass ratio (*Cresswell and Nelson, 2008*)

But their stability after the formation is not guaranteed
$\rightarrow$ *Pierens and Raymond 2014, Leleu et al. 2019*

Assuming the existence, the detection is challenging

- Mutual inclination $\rightarrow$ only one body transits.
- High mass ratio $\rightarrow$ only one body perturbs the radial velocity of the star.
- Large orbital period $\rightarrow$ only few transits per unit time & necessity of an almost zero mutual inclination.
- Low orbital period $\rightarrow$ strong tidal interaction with the host star

Identical co-orbital (same mass, radius, tidal parameters) are always destroyed by tides (*Rodriguez et al., 2013, using numerical simulations)*

$\rightarrow$ The libration amplitude increases until close encounters between the planets destroys the configuration.

No analytical work has been undertaken so far.
Do tides always disrupt the system ?
If so, on what timescale ?

Hamiltonian of the conservative problem

$$\mathcal{H} = \mathcal{H}_K(\Lambda_1, \Lambda_2) + H_P(\Lambda_1, \Lambda_2, \lambda_1, \lambda_2, x_1, x_2, \tilde{x}_1, \tilde{x}_2), \tag{1}$$

$$\mathcal{H}_K(\Lambda_1, \Lambda_2) = -\sum_{j=1}^{2} \frac{\beta_j^3 \mu_j^2}{2\Lambda_j^2}. \tag{2}$$

The semi-major axes stay close to a quantity denoted by $\bar{a}$.

$\rightarrow$ Expansion in the neighbourhood of the resonance

$$\mathbf{\Lambda}^\star = (\Lambda_1^\star, \Lambda_2^\star), \quad \text{with} \quad \Lambda_j^\star = m_j \sqrt{\mu_0 \bar{a}} \approx \Lambda_j. \tag{3}$$

Averaging over the fast orbital frequency

$$(\Lambda_1, \Lambda_2, \lambda_1, \lambda_2) \longmapsto (Z, Z_2, \phi, \phi_2) = (\Lambda_1 - \Lambda_1^\star,\ \Lambda_1 + \Lambda_2 - \Lambda_1^\star - \Lambda_2^\star,\ \lambda_1 - \lambda_2,\ \lambda_2).$$

$\rightarrow$ **Average over ϕ_2.**

Expansion of $\mathcal{H}_P$

$$\mathcal{H}_P = \sum_{n\geq 0} \mathcal{H}_{2n} \quad \text{where} \quad \mathcal{H}_{2n} = \sum_{|\mathbf{p}|=2n} \Psi_{\mathbf{p}}(\xi)\, X_1^{p_1} X_2^{p_2} \bar{X}_1^{\bar{p}_1} \bar{X}_2^{\bar{p}_2}\,. \tag{4}$$

$$\begin{aligned} & p_1 + p_2 = \bar{p}_1 + \bar{p}_2, \qquad \text{(D'alembert rule)}, \\ & \mathcal{H}_0 = \frac{m}{m_0}\left(\cos\xi - (2 - 2\cos\xi)^{-1/2}\right), \qquad \boxed{\xi = \lambda_1 - \lambda_2} \\ & \mathcal{H}_2 = \frac{1}{2}\frac{m}{m_0}\left\{A_h(\xi)\left(X_1\bar{X}_1 + X_2\bar{X}_2\right) + B_h(\xi)\, X_1\bar{X}_2 + \bar{B}_h(\xi)\,\bar{X}_1 X_2\right\}. \end{aligned} \tag{5}$$

X_j close to the eccentricity vector

$$X_j \approx e_j \exp(i\varpi_j). \tag{6}$$

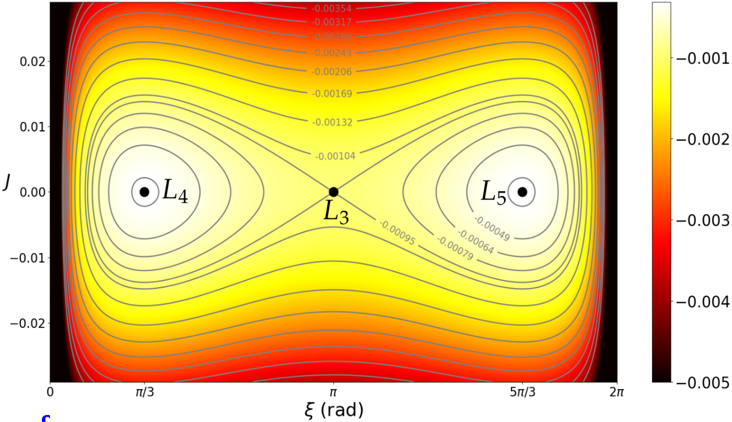
Invariance by rotation $\rightarrow$ D'alembert rule $\rightarrow \mathcal{H}_{2n+1} = 0$
$\rightarrow X_1 = X_2 = 0$ is a stable manifold of the space phase.

What about the dynamics at zero eccentricity ?

We study the one-degree-of-freedom Hamiltonian $\mathcal{H}_K + \mathcal{H}_0$

Space phase in the circular case

L_4 and L_5 are energy maximizers $\to$ L_4 and L_5 are tidally unstable.



Libration frequency

$\mathcal{O}\left(\varepsilon^{1/2}\right)$ in tadpole and early horseshoe. $\left(\sqrt{\frac{27\varepsilon}{4}} \text{ at } L_{4,5}\right)$

$\mathcal{O}\left(\varepsilon^{1/3}\right)$ in late horseshoe. $\varepsilon = (m_1 + m_2)/m_0$

Eccentric dynamics (Analytical approach of *Robutel & Pousse, 2013*)

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = -\frac{i}{m_0} \begin{pmatrix} m_2 A_h(\xi(t)) & m_2 \bar{B}_h(\xi(t)) \\ m_1 B_h(\xi(t)) & m_1 A_h(\xi(t)) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \tag{7}$$

Easy analytical solution if $\xi(t)$ is constant, that is, at $L_{4,5}$.

Two eigenvalues $ig_2 = 0$ and $ig_1 = i\frac{27\varepsilon}{8}$.

$\rightarrow$ **eigenvector associated with $g_2 = 0$** (a whole family of fixed points)

$$\begin{pmatrix} e^{i\pi/3} \\ 1 \end{pmatrix}, \text{ that is } e_1 = e_2 \text{ and } \varpi_1 - \varpi_2 = \pi/3 \ (0 \text{ in horseshoe}).$$

$\rightarrow$ **eigenvector associated with $g_1 \neq 0$** (a family of periodic orbits)

$$\begin{pmatrix} m_2 e^{i4\pi/3} \\ m_1 \end{pmatrix}, \text{ that is } \frac{e_1}{e_2} = \frac{m_2}{m_1} \text{ and } \varpi_1 - \varpi_2 = 4\pi/3 \ (\pi \text{ in horseshoe}).$$

Eccentric dynamics (Numerical approach of *Giuponne et al, 2010*)

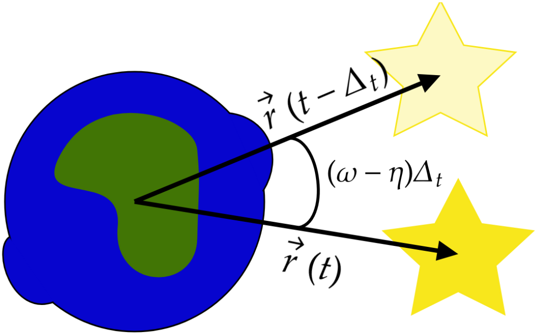
Reduction of the total angular momentum
$\rightarrow$ Only two degrees of freedom

- $\xi = \lambda_1 - \lambda_2$ and its associated action.
- $\Delta\varpi = \varpi_1 - \varpi_2$ and its associated action.

The Lagrange and anti-Lagrange eigen-directions now both appear as fixed points, not only Lagrange.

Problem for analytical work :
$\rightarrow$ The actions associated to ξ and $\Delta\varpi$ degenerate at zero eccentricity.
The approach of *Robutel & Pousse* is more suitable for analytical work.

The Mignard model, Mignard, 1979.

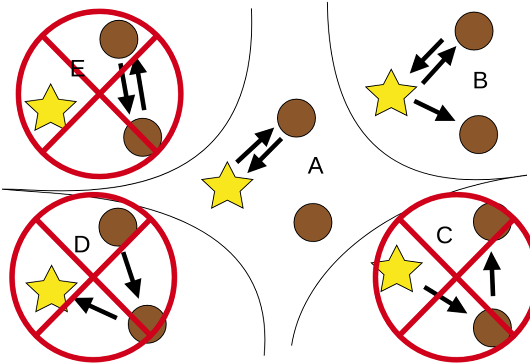


$$V(\mathbf{r}) = -\kappa_{2,j}\frac{\mathcal{G}m_i}{R_j}\left(\frac{R_j}{r}\right)^3\left(\frac{R_j}{r_i^{\star}}\right)^3 P_2\left(\cos S\right), \qquad \mathbf{r_i}^{\star} = \mathbf{r_i}\left(t-\Delta t_j\right).$$

Quality factor

$$Q_j^{-1}(\eta) = \sin(\eta\Delta t_j(\eta)) \approx \eta\Delta t_j(\eta). \tag{8}$$

Only the main contribution is retained : tides raised by the star on the planets and felt by the star.



Assuming $R \propto m^{1/3}$ and since $\kappa_{2,0} \approx 0.02$ while $\kappa_{2,j} \approx 0.5$

$$\frac{B}{A} = \frac{\kappa_{2,0} m_j^2 R_0^5}{\kappa_{2,j} m_0^2 R_j^5} = \frac{\kappa_{2,0}}{\kappa_{2,j}} \left(\frac{m_j}{m_0} \right)^{1/3} \ll 1. \tag{9}$$

Tidal (pseudo) Hamiltonian

$$\mathcal{H}_t = \mathcal{H}_t^1 + \mathcal{H}_t^2, \tag{10}$$

$$\mathcal{H}_t^j = -\kappa_{2,j}\mathcal{G}m_0^2 \frac{R_j^5}{r_j^3 r_j^{\star 3}} P_2\left(\cos S\right), \qquad S = \lambda_j - \lambda_j^\star - \left(\theta_j - \theta_j^\star\right). \tag{11}$$

Same transformations as in the conservative case, but no expansion over the semi major-axes.

$$\mathcal{H}_t^j = -q_j \frac{m_0}{m} \mathcal{R}_j^{-6} \mathcal{R}_j^{\star -6} \left\{ A_t^j + \Xi_2^j + \Xi_4^j + \mathcal{O}\left(|X_j|^6 \right) \right\},$$

$$A_t^j = \frac{1}{4} + \frac{3}{4} \cos 2 \left(\lambda_j - \lambda_j^\star - \theta_j + \theta_j^\star \right), \qquad q_j = \kappa_{2,j} \qoppa_j^5 = \kappa_{2,j} \left(\frac{R_j}{\bar{a}} \right)^5.$$

We define the dissipation rate [1]

$$\text{dissipation rate of planet } j : q_j / Q_j = \kappa_{2,j} \frac{R_j^5}{\bar{a}^5} \eta \Delta t_j. \tag{12}$$

We obtain an equation of the form [1]

$$\dot{\mathcal{X}} = F\left(\mathcal{X}\right), \qquad \mathcal{X} = {}^t\left(\omega_1, \omega_2, J, J_2, \xi, X_1, X_2 \right). \tag{13}$$

[1] *Couturier, Robutel & Correia, 2021*

Linearization of the differential system

in the vicinity of its equilibria $\rightarrow$ Small perturbation of L_4 & L_5.

Linear system

$$\dot{\mathcal{X}} = \left(\mathcal{Q}_0 + \mathcal{Q}_1\right)\mathcal{X}. \tag{14}$$

$\mathcal{Q}_0$ conservative part, $\mathcal{Q}_1$ dissipative part.

$$\mathcal{Q}_{0 \text{ or } 1} = \begin{pmatrix} * & 0_{5,2} \\ 0_{2,5} & * \end{pmatrix} \tag{15}$$

$\rightarrow$ Even in the dissipative case, the dynamics on (X_1, X_2) is uncoupled from the rest in the vicinity of the equilibrium.

Eigenvalues of $\mathcal{Q}_0$ (*Robutel & Pousse, 2013*)

$$\left\{0, 0, 0, i\nu, -i\nu, ig_1, ig_2\right\}, \qquad \nu = \sqrt{\frac{27\varepsilon}{4}}, \qquad g_1 = \frac{27\varepsilon}{8}, \qquad g_2 = 0. \tag{16}$$

Eigenvalues of $\mathcal{Q}_0 + \mathcal{Q}_1$ (*Couturier, Robutel & Correia, 2021*)

$$\left\{\lambda_1, \lambda_2, 0, \lambdabar, \bar{\lambdabar}, \lambda_{\mathsf{AL}}, \lambda_{\mathsf{L}}\right\}, \tag{17}$$

with (real parts are boxed)

$$\lambda_j = \boxed{-3\alpha_j^{-1}\frac{q_j}{Q_j}\varrho_j^{-2}\frac{m_0}{m_j} + 9\varepsilon^{-1}\frac{q_j}{Q_j}} < 0,$$

$$\lambdabar = \boxed{\frac{9}{2}\varepsilon^{-1}\left(\frac{m_1}{m_2}\frac{q_2}{Q_2} + \frac{m_2}{m_1}\frac{q_1}{Q_1}\right)} + i\nu\left[1 + 13\varepsilon^{-1}\left(\frac{m_1}{m_2}q_2 + \frac{m_2}{m_1}q_1\right)\right],$$

$$\lambda_{\mathsf{AL}} = \boxed{-\frac{21}{2}\varepsilon^{-1}\left(\frac{m_1}{m_2}\frac{q_2}{Q_2} + \frac{m_2}{m_1}\frac{q_1}{Q_1}\right)} + ig_1\left[1 + \frac{20}{9}\varepsilon^{-2}\left(\frac{m_1}{m_2}q_2 + \frac{m_2}{m_1}q_1\right)\right],$$

$$\lambda_{\mathsf{L}} = \boxed{-\frac{21}{2}\varepsilon^{-1}\left(\frac{q_1}{Q_1} + \frac{q_2}{Q_2}\right)} + \frac{15}{2}i\varepsilon^{-1}\left(q_1 + q_2\right).$$

Characteristic timescales of evolution (T is the orbital period)

$$\tau_{\mathsf{L}} = \frac{\varepsilon}{21\pi}\left(\frac{q_1}{Q_1}+\frac{q_2}{Q_2}\right)^{-1} T,$$
$$\tau_{\mathsf{AL}} = \frac{\varepsilon}{21\pi}\left(\frac{m_2}{m_1}\frac{q_1}{Q_1}+\frac{m_1}{m_2}\frac{q_2}{Q_2}\right)^{-1} T, \qquad (18)$$
$$\tau_{\mathsf{lib}} = \frac{7}{3}\tau_{\mathsf{AL}}.$$

Libration excitation [2]

$$\tau_{\mathsf{lib}} = \frac{1}{9\pi}\frac{\varepsilon}{q_1/Q_1+q_2/Q_2}\frac{\tau_{\mathsf{AL}}}{\tau_{\mathsf{L}}} T \propto \frac{\tau_{\mathsf{AL}}}{\tau_{\mathsf{L}}}\bar{a}^{6.5}. \qquad (19)$$

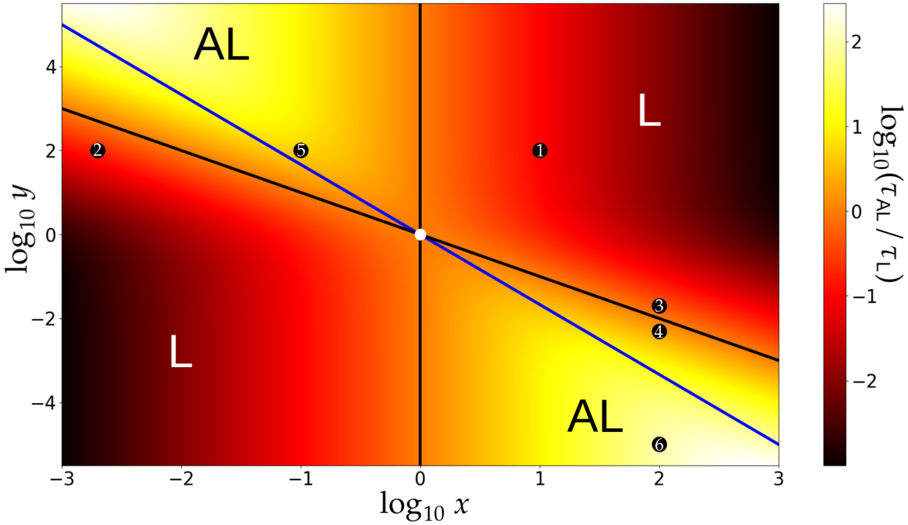
We define $x = m_1/m_2$ and $y = \frac{\text{dissipation rate}_2}{\text{dissipation rate}_1} = \frac{q_2 Q_1}{q_1 Q_2}$

Ratio between the eccentric damping timescales [2]

$$\frac{\tau_{\mathsf{AL}}}{\tau_{\mathsf{L}}} = \frac{x\,(1+y)}{1+yx^2}. \qquad (20)$$

[2] *Couturier, Robutel & Correia, 2021*

What are the anti-Lagrange-like systems ? $x = \frac{m_1}{m_2}$, $y = \frac{\text{dissipation rate}_2}{\text{dissipation rate}_1}$



Those with a very inequal mass repartition.

Two different sets of equations

- Top plot $\rightarrow$ the secular equations derived in this work.
- Bottom plot $\rightarrow$ A direct n-body model in cartesian coordinates.

$$\frac{d^2\boldsymbol{r}_1}{dt^2} = -\frac{\mu_1}{r_1^3}\boldsymbol{r}_1 + \mathcal{G}m_2\left(\frac{\boldsymbol{r}_2-\boldsymbol{r}_1}{|\boldsymbol{r}_2-\boldsymbol{r}_1|^3} - \frac{\boldsymbol{r}_2}{r_2^3}\right) + \frac{\boldsymbol{f}_1}{\beta_1} + \frac{\boldsymbol{f}_2}{m_0}\ ,$$

$$\frac{d^2\boldsymbol{r}_2}{dt^2} = -\frac{\mu_2}{r_2^3}\boldsymbol{r}_2 + \mathcal{G}m_1\left(\frac{\boldsymbol{r}_1-\boldsymbol{r}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3} - \frac{\boldsymbol{r}_1}{r_1^3}\right) + \frac{\boldsymbol{f}_2}{\beta_2} + \frac{\boldsymbol{f}_1}{m_0}\ ,$$

$$\frac{d^2\theta_i}{dt^2} = -\frac{(\boldsymbol{r}_i\times\boldsymbol{f}_i)\cdot\boldsymbol{k}}{C_i} = -3\frac{\kappa_{2,i}\mathcal{G}m_0^2R_i^3}{\alpha_i m_i r_i^8}\Delta t_i\left[\frac{d\theta_i}{dt}\,r_i^2 - \left(\boldsymbol{r}_i\times\frac{d\boldsymbol{r}_i}{dt}\right)\cdot\boldsymbol{k}\right]\ ,$$

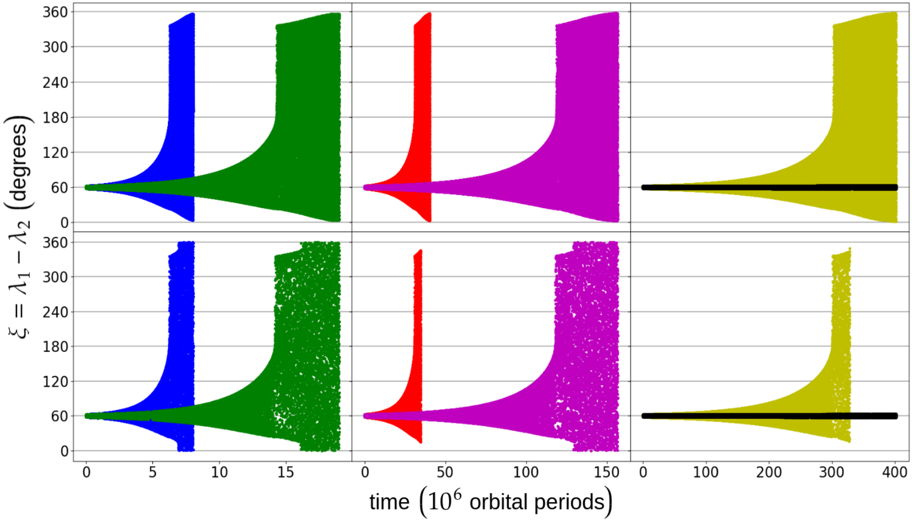
$$\boldsymbol{f}_i = -3\frac{\kappa_{2,i}\mathcal{G}m_0^2R_i^5}{r_i^8}\boldsymbol{r}_i$$

$$-\,3\frac{\kappa_{2,i}\mathcal{G}m_0^2R_i^5}{r_i^{10}}\Delta t_i\left[2\left(\boldsymbol{r}_i\cdot\frac{d\boldsymbol{r}_i}{dt}\right)\boldsymbol{r}_i + r_i^2\left(\frac{d\theta_i}{dt}\,\boldsymbol{r}_i\times\boldsymbol{k} + \frac{d\boldsymbol{r}_i}{dt}\right)\right]\ .$$

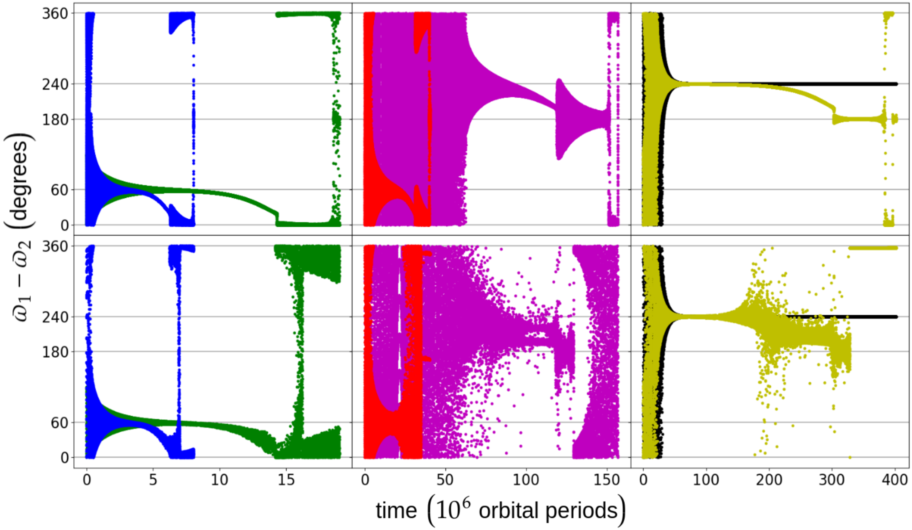
The secular equations

$$
\begin{aligned}
\dot{\vartheta}_j &= -3\alpha_j^{-1}\frac{m_0}{m_j}\qoppa_j^{-2}\frac{q_j}{Q_j}\mathcal{R}_j^{-12}\Big\{\vartheta_j + 3\left(1-\mathcal{R}_j\right) + h_2^j\mathcal{R}_j^{-1}X_j\bar{X}_j \\
&\qquad + h_4^j\mathcal{R}_j^{-2}X_j^2\bar{X}_j^2\Big\}, \\
\dot{J} &= -\frac{\partial\left(\mathcal{H}_0+\mathcal{H}_2+\mathcal{H}_4\right)}{\partial\xi} + (1-\delta)\,\dot{J}_2^1 - \delta\dot{J}_2^2, \\
\dot{J}_2 &= \dot{J}_2^1 + \dot{J}_2^2, \\
\dot{\xi} &= \frac{\partial\mathcal{H}_K}{\partial J} + 6q_1\frac{m_0}{m_1}\mathcal{R}_1^{-13}V_2\left(\mathcal{R}_1^{-1}X_1\bar{X}_1\right) - 6q_2\frac{m_0}{m_2}\mathcal{R}_2^{-13}V_2\left(\mathcal{R}_2^{-1}X_2\bar{X}_2\right), \\
\dot{X}_j &= -2i\frac{m}{m_j}\frac{\partial\left(\mathcal{H}_2+\mathcal{H}_4\right)}{\partial\bar{X}_j} \\
&\qquad - 3\frac{q_j}{Q_j}\frac{m_0}{m_j}\mathcal{R}_j^{-13}X_j\left\{p_2^j - \frac{5}{2}iQ_j + \frac{X_j\bar{X}_j}{\mathcal{R}_j}\left(p_4^j - \frac{65}{4}iQ_j\right)\right\},
\end{aligned}
$$

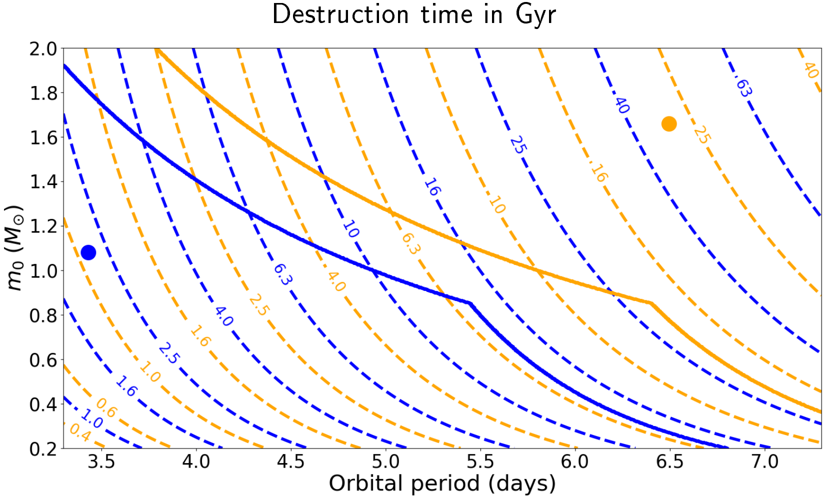
Libration amplitude $\rightarrow$ unbounded
Blue, green & red $\rightarrow$ short life.
Purple, yellow & black $\rightarrow$ long life.



Blue, green & red → settle into Lagrange.
Purple, yellow & black → settle into anti-Lagrange.



A tool to help with the detection of co-orbital exoplanets



Solid line $=\min$ (main sequence duration, universe age)

Orange dot $\rightarrow$ gas giant HD 102956 b

Blue dot $\rightarrow$ rocky planet HD 158259 c

Conclusion

Co-orbital exoplanets are always unstable under tides.

But they live long if :

- They orbit far away from the star $\rightarrow$ challenging detection
- The mass repartition is very inequal $\rightarrow$ challenging detection

We have a satisfactory explanation as to why no co-orbital planet has been detected so far, even though formation models predict their existence.

Much more details in the associated paper :
An analytical model for tidal evolution in co-orbital systems
10.1007/s10569-021-10032-w CM&DA

Thank you for your attention